6.003 Homework #6 Solutions

Problems

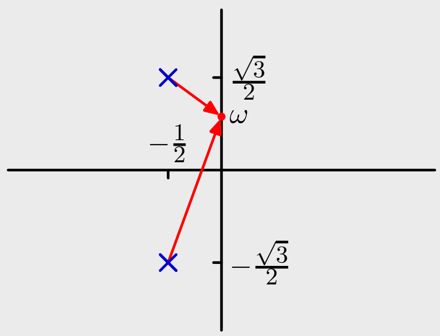
1. Maximum gain

For each of the following systems, find the frequency ω_m for which the magnitude of the gain is greatest.

a. $\dfrac{1}{1+s+s^2}$

$\omega_m =$ $\sqrt{\frac{1}{2}}$

This system has poles at $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.



We must minimize the product of the lengths of the two vectors. The product of the squared lengths is

$$\left(\left(\omega-\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)\left(\left(\omega+\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)=1-\omega^2+\omega^4$$

Minimize by taking derivative with respect to ω and setting to zero:

$$\frac{d}{d\omega}=-2\omega+4\omega^3=2\omega(2\omega^2-1)=0$$

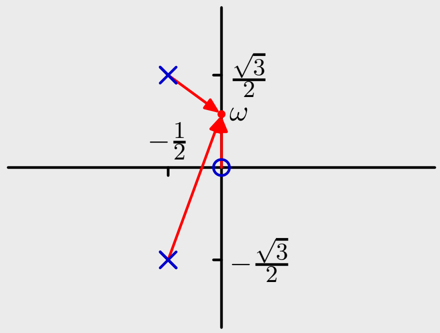
The solutions are $\omega = 0$ (which corresponds to a local maximum in the product of the lengths) or $\omega = \pm\sqrt{\frac{1}{2}}$. Thus the desired solution is

$$\omega=\sqrt{\frac{1}{2}}\,.$$

b. $\frac{s}{1+s+s^2}$

$\omega_m =$ 1

Now there is an added vector from the zero at $s = 0$.



The vector associated with the zero is in the numerator, while those associated with the poles are in the denominator. The squared quotient of lengths is

$$= \frac{\omega^2}{1-\omega^2+\omega^4}$$

Maximize by taking derivative with respect to ω and setting to zero:

$$\frac{d}{d\omega} = \frac{2\omega(\omega^4-1)}{(1-\omega+\omega^4)^2} = 0$$

The solutions are $\omega = 0$ (which corresponds to a minimum of the magnitude of the gain) or $\omega = \pm 1$ or $\omega = \pm j$. The desired solution is real and positive:

$$\omega = 1\,.$$

c. $\dfrac{s^2}{1+s+s^2}$

$\omega_m =$ $\sqrt{2}$

Now there are two added vectors from the zeros at $s = 0$. The squared quotient of lengths is

$$= \frac{\omega^4}{1-\omega^2+\omega^4}$$

Maximize by taking derivative with respect to ω and setting to zero:

$$\frac{d}{d\omega} = \frac{2\omega^3(2-\omega^2)}{(1-\omega+\omega^4)^2} = 0$$

The solutions are $\omega = 0$ (which again corresponds to a minimum) or $\omega = \pm\sqrt{2}$. The desired solution is

$$\omega = \sqrt{2}\,.$$

Compare the ω_m for these systems and make sure that you can explain qualitatively any similarities or differences.

The frequency of maximum gain increased from part a to b to c. The increase is because of the zero in part b and because of the two zeros in part c. The effect of the zero is to weight the gain by a factor that grows with frequency. This added weight pushes the maximum gain to progressively higher frequencies.

2. Phase

For a second-order system with poles at −1 and −4 (and no zeros), find the frequency at which the phase is −90°, using any method except for the vector method. Then illustrate and confirm that result using the vector method.

$\omega =$ 2

The system function has the form

$$H(s) = \frac{C}{(s+1)(s+4)} = \frac{C}{s^2+5s+4}\,.$$

Thus, the frequency response has the form

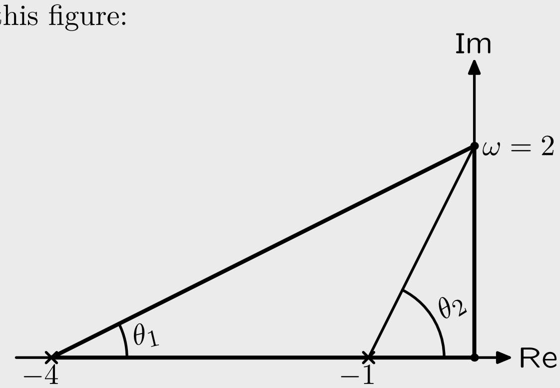
$$H(j\omega) = \frac{C}{(j\omega+1)(j\omega+4)} = \frac{C}{4-\omega^2+j5\omega}\,.$$

For very small values of ω, the denominator of $H(j\omega)$ is a small real number. As ω increases, the real part of the denominator decreases and the imaginary part increases. Assuming that C is positive and real, $H(j\omega)$ can only have an angle of −90° if the denominator has an angle of 90° — which means that the real part of the denominator must be zero:

$$\omega^2 = 4\,.$$

It follows that $\omega = 2$.

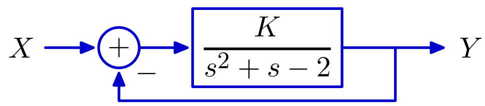
Next we illustrate this result using the vector method. The proposed −90° frequency is $\omega = 2$, so put in $s = j\omega_0 = 2j$. There, the phase of $(s+1)(s+4)$ is the sum of the two angles θ_1 and θ_2 in this figure:



The large, enclosing triangle (thick line) is a right triangle with side ratio 4 : 2. The smallest triangle is a right triangle with side ratio 1 : 2, which is the reverse of the side ratio for the large triangle. So the small triangle has the same shape as the large triangle but is rotated by 90 degrees (and is shrunk by a factor of 2). Thus θ_1 and θ_2 are the two non-right-angle angles of either triangle, and $\theta_1 + \theta_2 = 90°$. So the denominator has a 90° phase at $\omega = 2$, meaning that the overall system has a −90° phase at $\omega = \omega_0$, which is $\omega = 2$ as we wanted to show.

3. CT stability

Consider the following feedback system in which the box represents a causal LTI CT system that is represented by its system function.



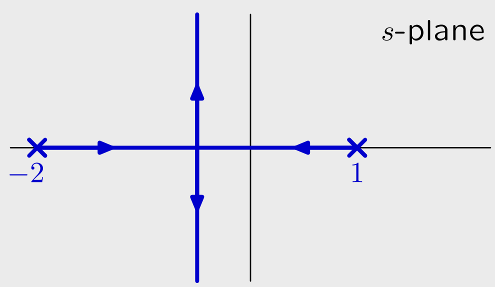
The closed-loop system response is

$$H(s) = \frac{\frac{K}{s^2+s-2}}{1+\frac{K}{s^2+s-2}} = \frac{K}{s^2+s+(K-2)}\,.$$

The closed-loop poles are at

$$s = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2-K}$$

as shown in the following figure, for $K > 0$.



a. Determine the range of K for which this feedback system is stable.

range of K: $K > 2$

This system is stable when the closed-loop poles are in the left half-plane, which is true for all $K > 2$.

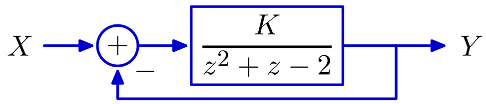
b. Determine the range of K for which this feedback system has real-valued poles.

range of K: $-\infty < K < \frac{9}{4}$

This system is has real valued poles if $-\infty < K < \dfrac{9}{4}$.

4. DT stability

Consider the following feedback system in which the box represents a causal LTI DT system that is represented by its system function.



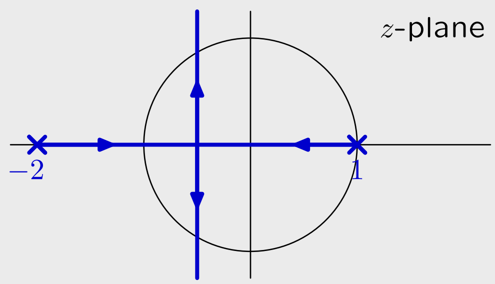
The closed-loop system response is

$$H(z) = \frac{\frac{K}{z^2+z-2}}{1+\frac{K}{z^2+z-2}} = \frac{K}{z^2+z+(K-2)}\,.$$

The closed-loop poles are at

$$z = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2-K}$$

as shown in the following figure, for $K > 0$.



a. Determine the range of K for which this feedback system is stable.

range of K: $2 < K < 3$

This system is stable when the closed-loop poles are inside the unit circle. To get the left pole inside the unit circle, K must be bigger than 2. To keep both complex-valued poles inside the unit circle, $K < 3$. In total $2 < K < 3$.

b. Determine the range of K for which this feedback system has real-valued poles.

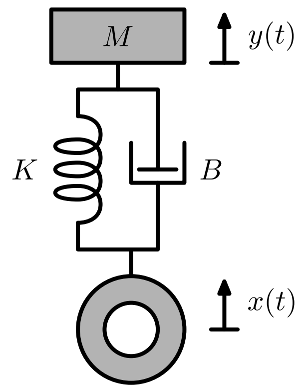
range of K: $-\infty < K < \frac{9}{4}$

This system is has real valued poles if $-\infty < K < \dfrac{9}{4}$.

Engineering Design Problems

5. Automotive suspension

Wheels are attached to an automobile through a suspension system that is designed to minimize the vibrations of the passenger compartment that result when traveling over bumpy terrain. The suspension system consists of a spring and shock absorber that are both compressed when the wheel passes over a bump, so that the sudden motion of the wheel is not directly transmitted to the passenger compartment. The spring generates a force to hold the passenger compartment at a desired distance above the surface of the road, and the shock absorber adds frictional damping. In this problem, you will determine how much damping is desireable by analyzing a simple model of an automobile's suspension system shown below.



The model consists of a mass M that represents the mass of the car, which is connected through a spring and dashpot to the wheel. The vertical displacement of the wheel from it's equilibrium position is taken as the input $x(t)$. The vertical displacement of the mass from it's equilibrium position is taken as the output $y(t)$. The spring is assumed to obey Hooke's law, so that the force it generates is a constant K times the amount that the spring is compressed relative to it's equilibrium compression. The shock absorber is assumed to generate a force that is a constant B times the velocity with which the shock absorber is compressed. Notice that by referring $x(t)$ and $y(t)$ to their equilibrium positions, the force due to gravity can be ignored. Assume that $M = 1$ and $K = 1$.

a. Determine the differential equation that relates the input $x(t)$ and output $y(t)$.

Start with Newton's law: $F = Ma$ where M represents the mass of the car and a represents the acceleration of the car. There are two important forces on the car: the spring force $K\big(x(t) - y(t)\big)$ and the force generated by the dashpot $B\big(\dot{x}(t) - \dot{y}(t)\big)$. Combining these, we find that

$$K\big(x(t) - y(t)\big) + B\big(\dot{x}(t) - \dot{y}(t)\big) = M\ddot{y}(t)$$

or equivalently

$$\ddot{y}(t) + B\dot{y}(t) + y(t) = x(t) + B\dot{x}(t)$$

since $M = K = 1$.

b. Determine and plot the impulse response of the system when $B = 0$. Based on this result, give a physical explanation of the problem that would result if there were no shock absorber in the system.

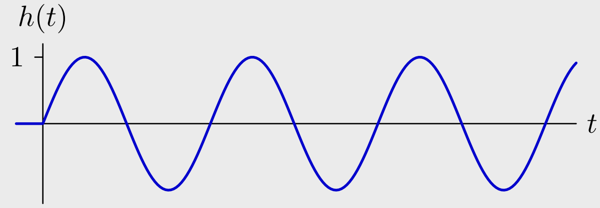
If $B = 0$ then the Laplace transform is:

$$H(s) = \frac{1}{s^2+1} = \frac{\frac{1}{2j}}{s-j} - \frac{\frac{1}{2j}}{s+j}\,.$$

The corresponding impulse response is

$$h(t) = \frac{1}{2j}\left(e^{jt} - e^{-jt}\right)u(t) = \sin(t)u(t)$$

as plotted below.



Without the shock absorber, there is no energy dissipation, and the car will oscillate forever after hitting a bump.

c. Determine an expression for the smallest positive damping constant B for which the poles of the system have real values. Sketch the impulse response of the system for this value of B. Based on this result, give a physical explanation of how the shock absorber improves performance of the suspension system.

In general, the Laplace transform is:

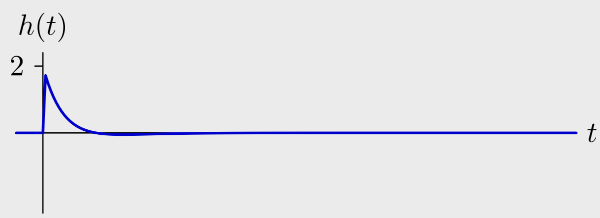
$$H(s) = \frac{Bs+1}{s^2+Bs+1},$$

The poles will have real values when $B \geq 2$. When $B = 2$ there are two poles at $s = -1$. Thus

$$H(s) = \frac{2s+1}{s^2+2s+1} = \frac{2}{s+1} - \frac{1}{(s+1)^2}$$

The corresponding impulse response is

$$h(t) = (2-t)e^{-t}u(t).$$



The shock absorber improves the system by damping out the oscillations that would otherwise result.

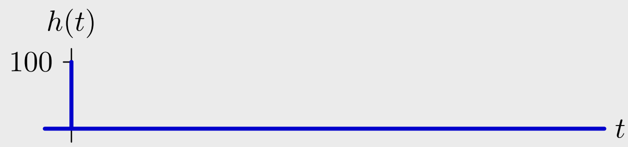
d. Consider what would happen if B were very large. Sketch the impulse response for the system if $B = 100$. Describe how this response might be less desireable than that in part c. Provide a physical explanation for how a stiff shock absorber can degrade system performance.

If $B = 100$ then the poles are approximately $s = -100$ and $s = -0.01$,

$$H(s) = \frac{100s+1}{s^2+100s+1} \approx \frac{100s+1}{(s+100)(s+0.01)} \approx \frac{100}{s+100}\,.$$

The corresponding impulse response is

$$h(t) = 100e^{-100t}u(t)\,.$$



The time constant for this response is 10 milliseconds, which is to short to be visible in the plot above. From the passenger's point of view, this response is very fast. Since the integral of $h(t)$ is

$$\int_{-\infty}^{\infty} h(t)dt = \int_{-\infty}^{\infty} 100e^{-100t}u(t)dt = 1$$

It follows that $h(t)$ is a good approximation to an impulse $\delta(t)$. Thus the output $y(t)$ is nearly equal to the input $x(t)$. The damping is so great that all vibrations of the wheels are transmitted to the car. The suspension system might just as well not be there!

6. Dial tones

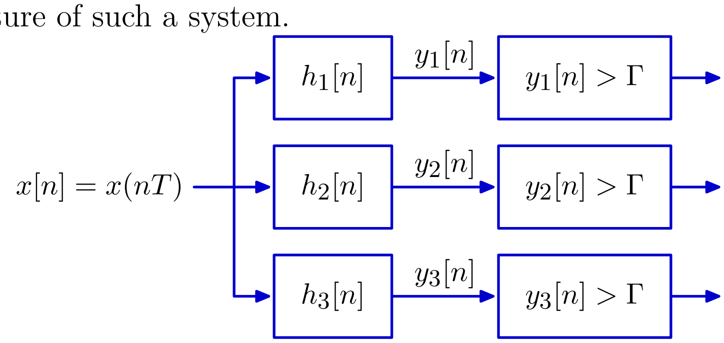
Pressing the buttons on a touch-tone phone generates tones that are used for dialing. Each button produces a pair of tones of the form

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$$

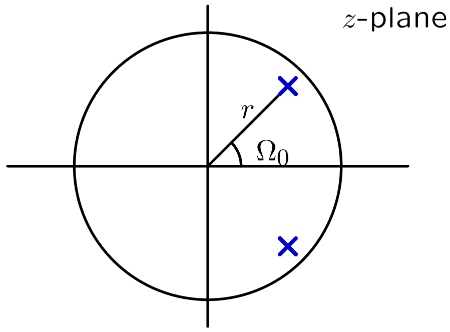
where f_1 and f_2 code the row and column of the button as shown in the following table.

	f_2[Hz]		
f_1[Hz]	1209	1336	1477
697	1	2	3
770	4	5	6
852	7	8	9
941	*	0	#

This problem concerns the design of a system to detect the row and column numbers that were pressed by analyzing the signal $x(t)$. The following block diagram illustrates the basic structure of such a system.



The input $x(t)$ is first sampled with $T = 10^{-4}$ seconds. The samples are then passed through LTI systems that generate intermediate signals so that $y_1[n]$ is large when a button in column 1 is pressed, $y_2[n]$ is large when a button in column 2 is pressed, and $y_3[n]$ is large when a button in column 3 is pressed. These intermediate signals are then passed through detectors that determine when the signals are bigger than a threshold value Γ. Your task is to design the LTI systems. Each should consist of a system with 2 poles of the form shown in the following pole-zero diagram.



Such systems can be simulated by finding the difference equation that corresponds to the system and then iteratively solving that difference equation.

a. Determine values of r and Ω_0 so that the $h_1[n]$ system generates a large response when the "1" key is pressed and a small response when the "2" or "'3" keys are pressed. Your solution should work not only when the input consists of a single key press but also when it consists of sequences of key presses (as when dialing a phone number). Submit hardcopies of your code to generate $y_1[n]$ along with a plot of $y_1[n]$.

For this problem consider the following input signal:

$$x[n] = \cos(\Omega_0 n) = \frac{\exp(j\Omega_0 n) + \exp(-j\Omega_0 n)}{2} = \frac{1}{2}\left\{\exp(j\Omega_0)^n + \exp(-j\Omega_0)^n\right\}.$$

This signal consists of eigenfunctions whose bases are $\exp(j\Omega_0)$ and $\exp(-j\Omega_0)$. Both of these are points on the unit circle, with angles given by $\pm\Omega_0$.

Consider a system given by two poles at $z = r \cdot \exp(\pm j\Omega_0)$. The frequency response of this system will have its magnitude peak at frequency $\pm\Omega_0$. Hence the signal $x[n]$ given above will be amplified more when $\Omega_0 \approx \Omega_0$. We use this to discriminate the three possible tones, by setting $\Omega_0 = 2\pi 0.1209$.

Given a system with two poles, suppose the two poles are at $z_A = r\exp(j\Omega_0)$ and $z_B = r\exp(-j\Omega_0)$. Then we can write

$$\frac{Y(z)}{X(z)} = \frac{1}{(z - z_A)(z - z_B)} = \frac{1}{z^2 - (z_A + z_B)z + z_A z_B},$$

or in other words, as a difference equation:

$$z^2 Y(z) - (z_A + z_B)zY(z) + z_A z_B Y(z) = X(z),$$
$$y[n+2] - (z_A + z_B)y[n+1] + z_A z_B y[n] = x[n].$$

```
r = 0.99;
w_0 = 2*pi*0.1209;
zA = r*exp(j*w_0);
zB = r*exp(-j*w_0);
y = zeros(1,3000);
for n = 1:(length(y)-2)
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```
    y(n+2) = (zA+zB)*y(n+1) - (zA*zB)*y(n) + x(n);
end
```

b. Describe how the choice of Ω_0 affects the output signal $y_1[n]$.

The closer Ω_0 is to $2\pi 0.1209$, the larger the magnitude at the output corresponding to that tone is. If Ω_0 approaches one of the other tones, then the magnitude at the output corresponding to those respective tones are larger.

c. Describe how the choice of r affects the output signal $y_1[n]$. In particular, what limits the maximum acceptable value of r? Also, what limits the minimum acceptable value of r?

In order to increase the relative amplification of tone 1, we wish to choose r to be close to 1 but without making the system unstable. Choosing r to be between 0.9 and 0.99 gives good discrimination of the tones without incurring instability. This is shown in the following figure. When r is chosen to be too close to 1, or greater than 1, the system becomes unstable.