6.003 Homework #4 Solutions

Problems

1. Laplace Transforms

Determine the Laplace transforms (including the regions of convergence) of each of the following signals:

a. $x_1(t) = e^{-2(t-3)}u(t-3)$

$X_1 = \boxed{\dfrac{e^{-3s}}{s+2}}$

ROC: $\boxed{\text{Re}(s) > -2}$

$$X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-2(t-3)}u(t-3)e^{-st}dt = e^6 \int_3^{\infty} e^{-(s+2)t}dt$$

$$= e^6 \left. \frac{e^{-(s+2)t}}{-(s+2)} \right|_3^{\infty} = \frac{e^{-3s}}{s+2}; \quad \text{Re}(s) > -2$$

b. $x_2(t) = (1-(1-t)e^{-3t})u(t)$

$X_2 = \boxed{\dfrac{4s+9}{s(s+3)^2}}$

ROC: $\boxed{\mathrm{Re}(s) > 0}$

Treat this as the sum of 3 signals: $x_2(t) = x_{2a}(t) + x_{2b}(t) + x_{2c}(t)$, where $x_{2a}(t) = u(t)$, $x_{2b}(t) = -e^{-3t}u(t)$, and $x_{2c}(t) = te^{-3t}u(t)$.

$$X_{2a}(s) = \int_0^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}\ ;\quad \mathrm{Re}(s) > 0$$

$$X_{2b}(s) = \int_0^\infty -e^{-3t}e^{-st}dt = \left.\frac{-e^{-(s+3)t}}{-(s+3)}\right|_0^\infty = -\frac{1}{s+3}\ ;\quad \mathrm{Re}(s) > -3$$

$$X_{2c}(s) = \int_0^\infty te^{-3t}e^{-st}dt = \left.t\frac{e^{-(s+3)t}}{-(s+3)}\right|_0^\infty - \int_0^\infty \frac{e^{-(s+3)t}}{-(s+3)}dt = \frac{1}{(s+3)^2}\ ;\ \mathrm{Re}(s) > -3$$

The Laplace transform of a sum is the sum of the Laplace transforms,

$$X_2(s) = \frac{1}{s} - \frac{1}{s+3} + \frac{1}{(s+3)^2} = \frac{4s+9}{s(s+3)^2}$$

and the region of convergence is the intersection of the 3 regions: $\mathrm{Re}(s) > 0$.

An alternative way to find the transform of $x_{2c} = te^{-3t}u(t)$ is to realize that multiplying a time function by $-t$ is equivalent to differentiating its transform by s:

$$\frac{dX(s)}{ds} = \frac{d}{ds}\left(\int x(t)e^{-st}dt\right) = \int -tx(t)e^{-st}dt$$

It follows that

$$X_{2c} = \frac{d}{ds}X_{2b} = \frac{d}{ds}\left(-\frac{1}{s+3}\right) = \frac{1}{(s+3)^2}$$

c. $x_3(t) = |t|e^{-|t|}$

$X_3 =$ $\boxed{\dfrac{2(s^2+1)}{(s+1)^2(s-1)^2}}$

ROC: $\boxed{-1 \ < \ \mathrm{Re}(s) \ < \ 1}$

The signal $x_3(t)$ can be written as $te^{-t}u(t) - te^{t}u(-t)$. The transform of the first is

$$\frac{1}{(s+1)^2} \;;\quad \mathrm{Re}(s) > -1$$

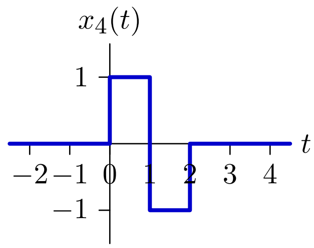
and the transform of the second is

$$\frac{1}{(s-1)^2} \;;\quad \mathrm{Re}(s) < 1.$$

Therefore

$$X_3(s) = \frac{1}{(s+1)^2} + \frac{1}{(s-1)^2} = \frac{2(s^2+1)}{(s+1)^2(s-1)^2} \;;\quad -1 < \mathrm{Re}(s) < 1.$$

d.



$X_4 =$ $\dfrac{e^{-2s} - 2e^{-s} + 1}{s}$

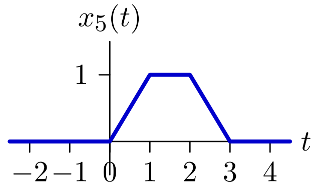
ROC: all s

$$X_4(s) = \int_{-\infty}^{\infty} x_4(t)e^{-st}dt = \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^1 - \left.\frac{e^{-st}}{-s}\right|_1^2$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s} = \frac{e^{-2s} - 2e^{-s} + 1}{s}$$

These integrals all converge for all values of s. Therefore the region of convergence is the entire s-plane.

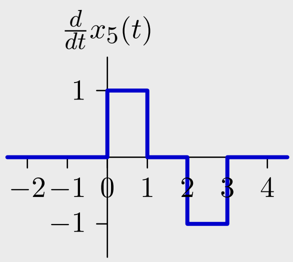
e.



$X_5 =$ $\dfrac{1 - e^{-s} - e^{-2s} + e^{-3s}}{s^2}$

ROC: all s

This is not so easy to do directly from the definition, since this leads to integrals of t times e^{-st}. The result can be integrated by parts, but it is messy. An easier way is to realize that the derivative of $x_5(t)$ is simple:



This function can be written as $u(t) - u(t-1) - u(t-2) + u(t-3)$ and therefore has a Laplace transform equal to

$$\frac{1}{s}(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

Since $x_5(t)$ is the integral of it's derivative, the Laplace transform of $x_5(t)$ is $1/s$ times the Laplace transform of its derivative:

$$X_5(s) = \frac{1}{s^2}(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

The region of convergence includes the whole s-plane since $x_5(t)$ has finite duration.

2. Inverse Laplace transforms

Determine all possible signals with Laplace transforms of the following forms. For each signal, indicate a closed-form solution as well as the region of time t for which the closed-form solution is valid. Three boxes are given for each part. If fewer than three solutions exist for a given part, indicate **none** in the extra boxes.

a. $X_1(s) = \dfrac{s+2}{(s+1)^2}$

	$t < 0$	$t > 0$
$x_1(t) =$	$-(1+t)e^{-t}$	0
or	0	$(1+t)e^{-t}$
or	none	none

We can expand $X_1(s)$ using partial fractions as

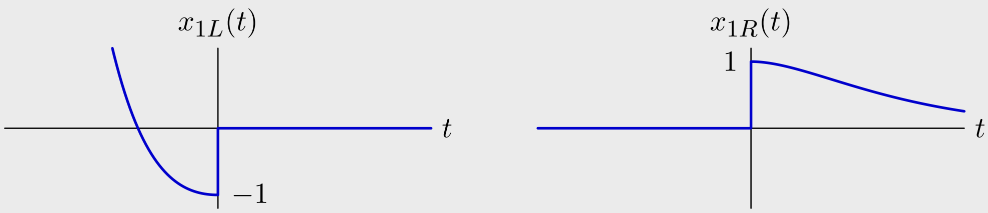
$$X_1(s) = \frac{s+2}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

Because both poles are at $s = -1$, there are just two possible regions of convergence: $s > -1$ and $s < -1$. For the left-sided region,

$$x_{1L}(t) = -(1+t)e^{-t}u(-t).$$

For the right-sided region,

$$x_{1R}(t) = (1+t)e^{-t}u(t).$$



b. $X_2(s) = \dfrac{1}{s^2(s-1)}$

	$t<0$	$t>0$
$x_2(t) =$	$-e^t + 1 + t$	0
or	$-e^t$	$-1-t$
or	0	$e^t - 1 - t$

Using partial fractions,

$$X_2(s) = \frac{1}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}.$$

The two poles at $s=0$ and the pole at $s=1$ break the s-plane into 3 possible regions of convergence: $s<0$, $0<s<1$, and $s>1$. For $s<0$, all of the terms are left-sided, so

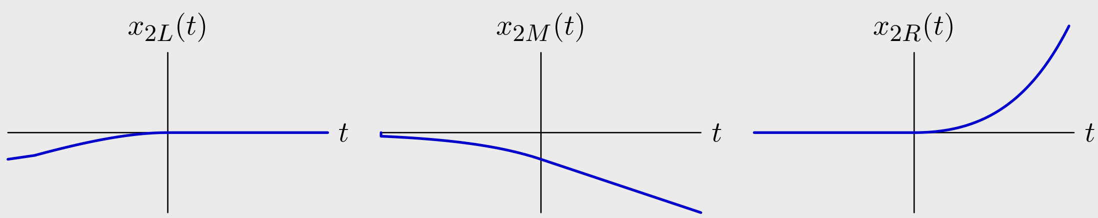
$$x_{2L}(t) = -e^t u(-t) + u(-t) + tu(-t).$$

For $0<s<1$, exponential term is left-sided and the others are right-sided, so

$$x_{2M}(t) = -e^t u(-t) - u(t) - tu(t).$$

For $s>1$, all of the terms are right-sided, so

$$x_{2R}(t) = e^t u(t) - u(t) - tu(t).$$



c. $X_3(s) = \dfrac{s+1}{s^2+2s+2}$

	$t<0$	$t>0$
$x_3(t) =$	$-e^{-t}\cos t$	0
or	0	$e^{-t}\cos t$
or	none	none

The factors of the denominator of $X_3(s)$ are complex-valued. Nevertheless, partial fractions still work.

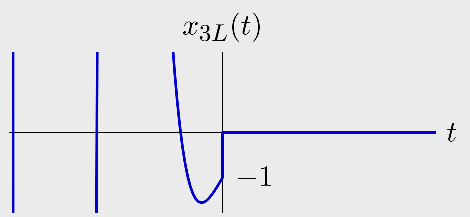
$$X_3(s) = \frac{s+1}{s^2+2s+2} = \frac{1/2}{s+1+j} + \frac{1/2}{s+1-j}.$$

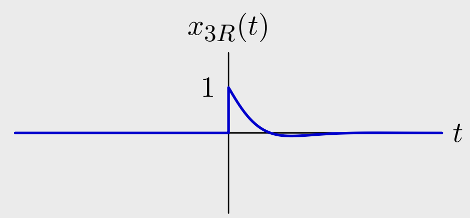
Both of these poles have the same real part. Therefore, there are two possible regions of convergence: $s > -1$ and $s < -1$. Both terms are left-sided for $s < -1$, so

$$x_{3L}(t) = -e^{-t}\cos(t)u(-t).$$

Both terms are right-sided for $s > -1$, so

$$x_{3R}(t) = \frac{1}{2}e^{-(1+j)t}u(t) + \frac{1}{2}e^{-(1-j)t}u(t) = e^{-t}\cos(t)u(t).$$





The values of $|x_{3L}(t)|$ are so large that regions in its plot are clipped.

d. $X_4(s) = \left(\frac{1-e^{-s}}{s}\right)^2$

	$t<0$	$0<t<1$	$1<t<2$	$t>2$
$x_4(t) =$	0	t	$2-t$	0
or	none	none	none	none
or	none	none	none	none

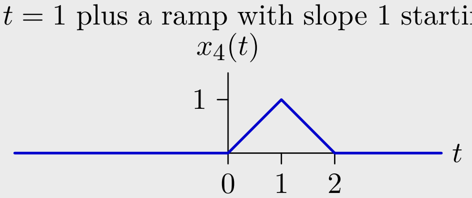
This expands to

$$X_4(s) = \left(\frac{1-e^{-s}}{s}\right)^2 = \frac{1-2e^{-s}+e^{-2s}}{s^2}.$$

Laplace transforms of the for $\frac{1}{s^n}$ correspond to derivatives of sigularity functions.

$$\begin{aligned} \delta(t) \quad &\leftrightarrow \quad 1 \\ u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \quad &\leftrightarrow \quad \frac{1}{s} \\ tu(t) = \int_{-\infty}^{t} u(\tau)d\tau \quad &\leftrightarrow \quad \frac{1}{s^2} \end{aligned}$$

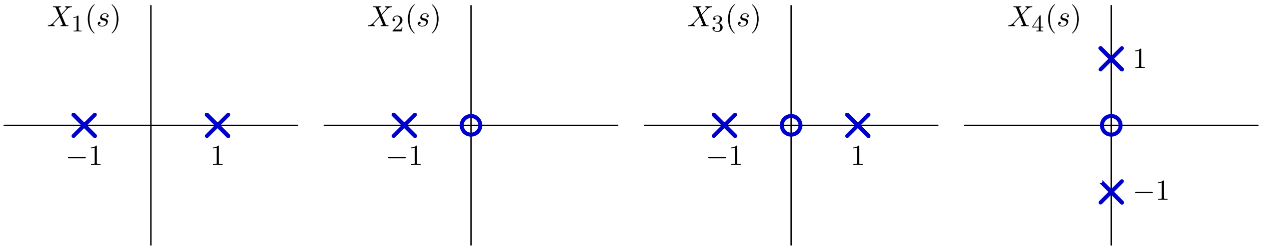
Furthermore, the exponential terms represent delay. If $x(t) \leftrightarrow X(s)$ then $x(t-\tau) \leftrightarrow e^{-s\tau}X(s)$. Therefore, $x_4(t)$ is a ramp with slope 1 starting at $t=0$ plus a ramp with slope -2 starting at $t=1$ plus a ramp with slope 1 starting at $t=2$, as shown below.



Since $X_4(s)$ converges everywhere in s, there is a single region of convergence, so the plot above shows the only inverse transform of $X_4(s)$. [Uniqueness requires that if multiple signals have Laplace transforms of the same form, then the transform must have multiple, non-overlapping regions of convergence: one for each distinct signal.]

3. Symmetry

Determine which of the following pole-zero diagrams could represent Laplace transforms of even functions of time.



Enter a subset of the numbers 1 through 4 (separated by spaces) to represent $X_1(s)$ through $X_4(s)$ in the answer box below. If none of $X_1(s)$ through $X_4(s)$ apply, enter **none**.

1 and/or 2 and/or 3 and/or 4 or **none**: 1

Explain.

If $x(t)$ is an even function of time, then the corresponding Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(-t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{st}dt = X(-s)$$

must be symmetric about the point $s = 0$. Thus the pole-zero diagram must also be symmetric about $s = 0$. This means that $X_2(s)$ cannot correspond to an even function of time.

For $X(s)$ to be symmetric about $s = 0$, then the region of convergence must be symmetric about the $j\omega$ axis. Symmetry is not possible for $X_4(s)$, which must be either right-sided or left-sided but not both.

Both $X_1(s)$ and $X_3(s)$ have similar partial fraction expansions:

$$\frac{A}{s+1} + \frac{B}{s-1}.$$

To get a zero at zero (as in $X_3(s)$), $A = B$. The corresponding time function is

$$x_3(t) = A\left(e^{-t}u(t) - e^{t}u(-t)\right)$$

which is an odd function of time for all values of A. To get no finite zeros (as in $X_1(s)$), $A = -B$. The corresponding time function is

$$x_1(t) = A\left(e^{-t}u(t) + e^{t}u(-t)\right)$$

which is an even function of time for all values of A. Thus, only $x_1(t)$ is an even function of t.

This example illustrates that symmetry of the pole-zero diagram is necessary but not sufficient for symmetry of the corresponding time function.

4. Initial and final values (from "Circuits, Signals, and Systems" by Siebert)

a. Use the initial and final value theorems (where applicable) to find $x(0)$ and $x(\infty)$ for the signals with the following Laplace transforms:

1. $\dfrac{1-e^{-sT}}{s}$
2. $\dfrac{1}{s}e^{-sT}$
3. $\dfrac{1}{s(s+1)^2}$
4. $\dfrac{1}{s^2(s+1)}$
5. $\dfrac{1}{s^2+1}$
6. $\dfrac{(s+1)^2-1}{[(s+1)^2+1]^2}$

Assume that the regions of convergence include $\text{Re}(s) > 0$.

b. Find the inverse Laplace transforms for each of the previous parts and show that the time waveforms and initial and final values agree.

Enter your answers in the boxes below. If the initial or final value theorem cannot be applied, enter **X**.

	$x(0)$	$x(\infty)$
1.	1	0
2.	0	1
3.	0	1
4.	0	X
5.	0	X
6.	0	0

$X(s)$	$x(t)$	$x(0)$	$x(\infty)$
$\dfrac{1-e^{-sT}}{s}$	$u(t)-u(t-T)$	1	0
$\dfrac{1}{s}e^{-sT}$	$u(t-T)$	0	1
$\dfrac{1}{s(s+1)^2} = \dfrac{1}{s} - \dfrac{1}{s+1} - \dfrac{1}{(s+1)^2}$	$(1-(1+t)e^{-t})u(t)$	0	1
$\dfrac{1}{s^2(s+1)} = -\dfrac{1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s+1}$	$(t-1+e^{-t})u(t)$	0	X[1]
$\dfrac{1}{s^2+1} = \dfrac{j/2}{s+j} - \dfrac{j/2}{s-j}$	$\sin(t)u(t)$	0	X[2]
$\dfrac{(s+1)^2-1}{[(s+1)^2+1]^2} = \dfrac{1/2}{((s+1)+j)^2} + \dfrac{1/2}{((s+1)-j)^2}$	$te^{-t}\cos(t)u(t)$	0	0

[1] The final value theorem does not apply because $x(t)$ does not approach a limit as $t \to \infty$. Rather, the function grows with t, because there is a pole at 0.

[2] The final value theorem does not apply because $x(t)$ does note approach a limit as $t \to \infty$. Rather, the function oscillates because $sX(s)$ has poles on the $j\omega$ axis

Engineering Design Problems

5. Impulse response

Sketch a block diagram for a CT system with impulse response

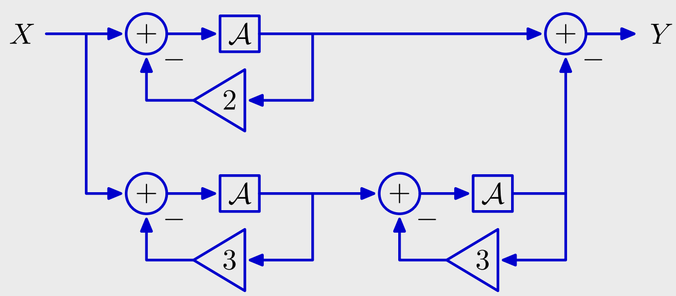
$$h(t) = \left(1 - te^{-t}\right) e^{-2t} u(t)\,.$$

The block diagram should contain only adders, gains, and integrators.

Take the Laplace transform to find that

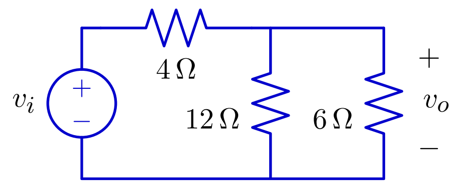
$$H(s) = \frac{1}{s+2} - \left(\frac{1}{s+3}\right)^2 .$$

Then substitute $\frac{1}{s} \rightarrow \mathcal{A}$ and write as a block diagram.



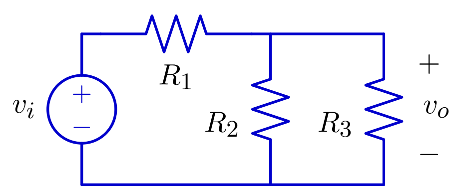
6. Impedance Method

a. Determine the output voltage of the following circuit, using series or parallel resistor combinations and/or voltage or current dividers.



The parallel combination of $12\,\Omega$ and $6\,\Omega$ is $\frac{12\times 6}{12+6} = 4\,\Omega$. This $4\,\Omega$ "equivalent" resistance forms a voltage divider with the top resistor. The voltage divider halves the input voltage. Thus $v_o = \frac{1}{2}v_i$.

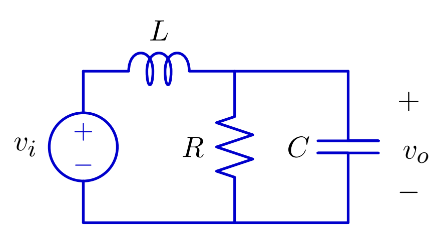
b. Generalize the result from part a for arbitrary resistor values, and determine an expression for the resulting ratio $\dfrac{v_o}{v_i}$.



Now the equivalent resistance is $\frac{R_2R_3}{R_2+R_3}$. In combination with R_1, the voltage divider generates

$$\frac{v_o}{v_i} = \frac{\frac{R_2R_3}{R_2+R_3}}{R_1 + \frac{R_2R_3}{R_2+R_3}} = \frac{R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

c. Consider the following circuit.



Determine a differential equation that relates v_i to v_C as follows. First, determine relations among the element currents (i_R, i_L, and i_C) and element voltages (v_R, v_L, and v_C) using KVL and KCL. Second, relate each element voltage to the corresponding element current using the constitutive relation for the element: i.e., $v_R = Ri_R$, $i_C = C\frac{dv_C}{dt}$, and $v_L = L\frac{di_L}{dt}$. Finally, solve your equations to find a single equation with terms that involve v_i, v_o, and derivatives of v_i and v_o.

KCL: $i_L = i_R + i_C$.
KVL: $v_R = v_C$ and $v_i = v_L + v_R$.
parts: $v_R = Ri_R$, $i_C = C\frac{dv_C}{dt}$, and $v_L = L\frac{di_L}{dt}$.
Eliminate the KVL equations by substituting $v_L \to v_i - v_C$ and $v_C \to v_R$ into the remaining equations. Then eliminate the parts equations by substituting $\dfrac{di_L}{dt} \to \dfrac{v_L}{L}$,

$\frac{di_R}{dt} \to \frac{1}{R}\frac{dv_R}{dt}$, and $\frac{di_C}{dt} \to C\frac{d^2v_C}{dt^2}$ into the KCL equation (after differentiating each term by t. The resulting equation is

$$\frac{v_i - v_C}{L} = \frac{1}{R}\frac{dv_C}{dt} + C\frac{d^2v_C}{dt^2}$$

which can be simplifed to

$$v_i = v_C + \frac{L}{R}\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} = v_o + \frac{L}{R}\frac{dv_o}{dt} + LC\frac{d^2v_o}{dt^2}$$

d. Determine the system function $H(s) = \frac{V_o(s)}{V_i(s)}$, based on the Laplace transform of your answer to the previous part.

$$V_i = (1 + s\frac{L}{R} + s^2LC)V_o$$

$$\frac{V_o}{V_i} = \frac{1}{1 + s\frac{L}{R} + s^2LC}$$

e. Substitute $R_1 \to sL$, $R_2 \to R$, and $R_3 \to \frac{1}{sC}$ into your result from part b.

$$\frac{R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3} = \frac{R\frac{1}{sC}}{sLR + sL\frac{1}{sC} + R\frac{1}{sC}} = \frac{1}{s^2LC + s\frac{L}{R} + 1}$$

Compare this new expression to your result from part d.

The substitute gives the same expression that we found in part d.

f. The impedance of an electrical element is a function of s that can be analyzed using rules that are quite similar to those for resistances. Explain the basis of this method.

The impedance method is equivalent to taking the Laplace transforms of all of the normal circuit equations. KVL and KCL are not changed, because the Laplace transform is linear, i.e., the Laplace transform of a sum (of voltages or currents) is simply the sum of the Laplace transforms. Ohm's law for a resistor is also unchanged: if $v_R = Ri_R$ then $V_R = RI_R$. However the constitutive laws for inductors and capacitors change: $v_L = L\frac{di_L}{dt}$ becomes $V_L = sLI_L$ and $i_c = C\frac{dv_C}{dt}$ becomes $I_C = sCV_C$. These latter laws have the same form as Ohm's law if we interpret impedances as R (for resistors), sL (for inductors), and $\frac{1}{sC}$ (for capacitors).