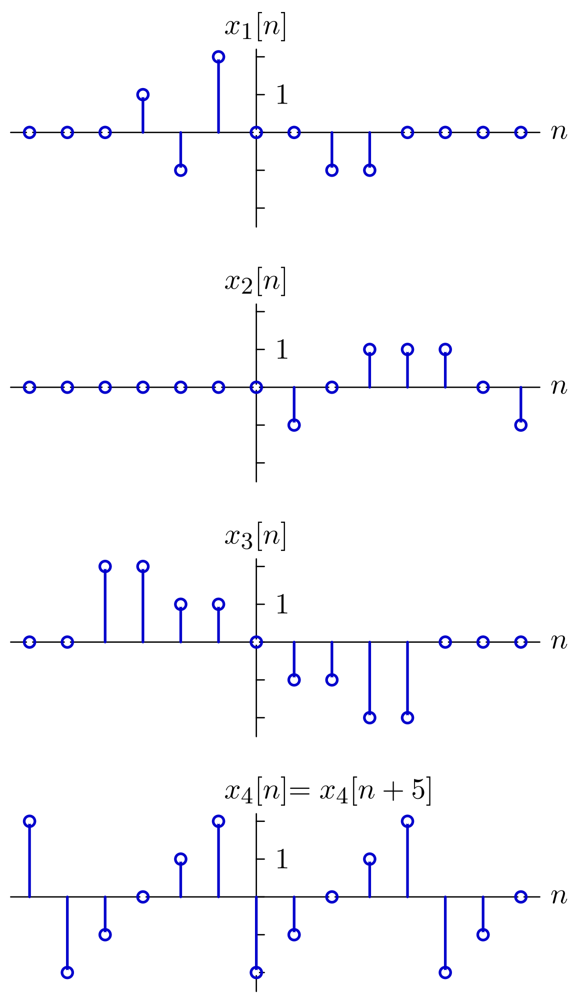
6.003 Homework #12 Solutions

Problems

1. Which are True?

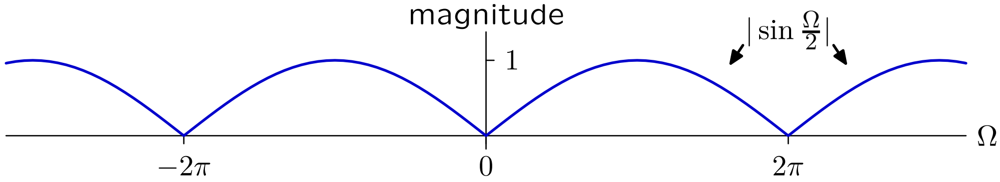
For each of the DT signals $x_1[n]$ through $x_4[n]$ (below), determine whether the conditions listed in the following table are satisfied, and answer **T** for true or **F** for false.

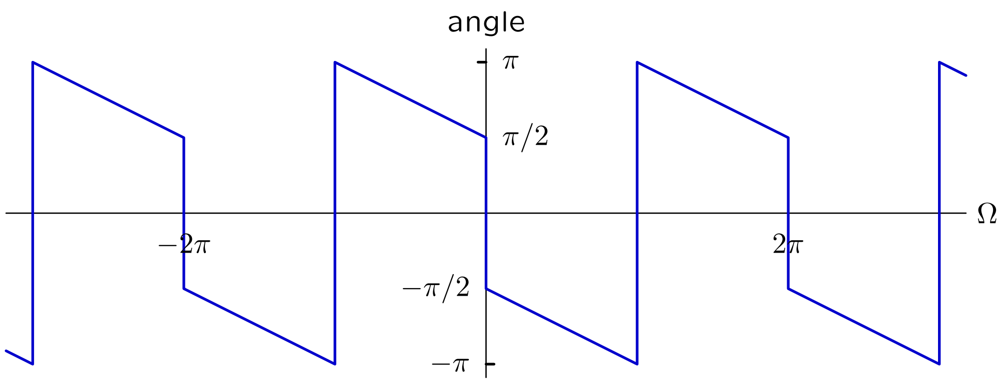
	$x_1[n]$	$x_2[n]$	$x_3[n]$	$x_4[n]$
$X(e^{j0}) = 0$	T	F	T	T
$\int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega = 0$	T	T	T	F
$X(e^{j\Omega})$ is purely imaginary	F	F	T	F
$e^{jk\Omega}X(e^{j\Omega})$ is purely real for some integer k	F	T	F	F



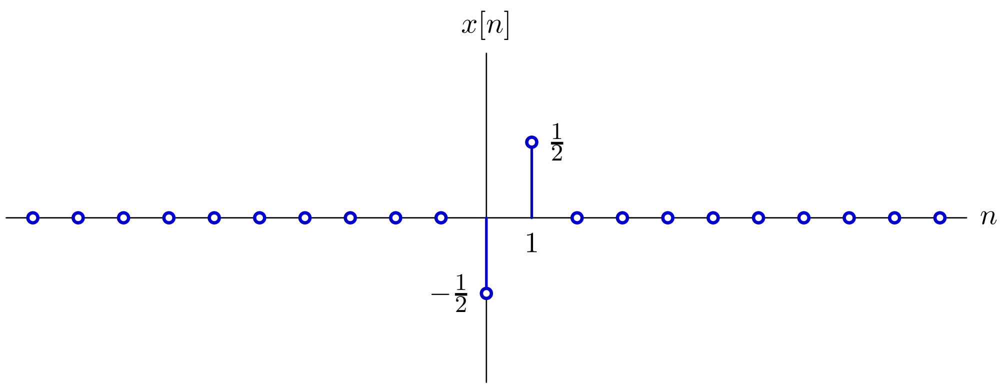
2. Inverse Fourier

The magnitude and angle of the Fourier transform of $x[n]$ are shown below.





Determine $x[n]$.



$$X(e^{j\Omega}) = -j\,\sin\frac{\Omega}{2}\,e^{-j\Omega/2} = -j\left(\frac{e^{j\Omega/2}-e^{-j\Omega/2}}{j2}\right)e^{-j\Omega/2} = \frac{1}{2}e^{-j\Omega} - \frac{1}{2}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{4\pi}\int_{2\pi}\left(e^{-j\Omega}-1\right)e^{j\Omega n}d\Omega = \frac{1}{4\pi}\int_{2\pi}\left(e^{j\Omega(n-1)}-e^{j\Omega n}\right)d\Omega$$

$$= \begin{cases} -1/2 & n=0 \\ 1/2 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

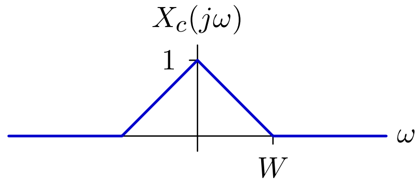
Engineering Design Problem

3. Sampling with alternating impulses

A CT signal $x_c(t)$ is converted to a DT signal $x_d[n]$ as follows:

$$x_d[n] = \begin{cases} x_c(nT) & n \text{ even} \\ -x_c(nT) & n \text{ odd} \end{cases}$$

a. Assume that the Fourier transform of $x_c(t)$ is $X_c(j\omega)$ shown below.



Determine the DT Fourier transform $X_d(e^{j\Omega})$ of $x_d[n]$.

Let $x_e[n] = (-1)^n x_d[n] = x_c(nT)$. Then

$$x_e[n] = \frac{1}{2\pi}\int_{2\pi} X_e(e^{j\Omega})e^{j\Omega n}d\Omega = (-1)^n x_d[n] = (-1)^n \frac{1}{2\pi}\int_{2\pi} X_d(e^{j\Omega})e^{j\Omega n}d\Omega$$
$$= e^{-jn\pi}\frac{1}{2\pi}\int_{2\pi} X_d(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} X_d(e^{j\Omega})e^{j(\Omega-\pi)n}d\Omega$$
$$= \frac{1}{2\pi}\int_{2\pi} X_d(e^{j(\Omega'+\pi)})e^{j\Omega' n}d\Omega'$$

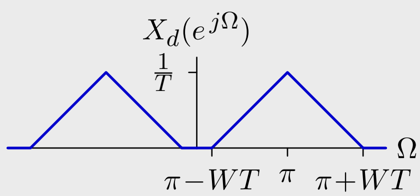
Thus $X_e(e^{j\Omega}) = X_d(e^{j(\Omega+\pi)})$. Also

$$x_e[n] = \frac{1}{2\pi}\int_{2\pi} X_e(e^{j\Omega})e^{j\Omega n}d\Omega = x_c(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\omega)e^{j\omega nT}d\omega$$

Substituting $\omega = \frac{\Omega}{T}$ we see that $X_e(e^{j\Omega}) = \frac{1}{T}X_c(j\omega)\big|_{\omega=\frac{\Omega}{T}}$ provided $X_c(j\omega) = 0$ for $|\omega| > \frac{\pi}{T}$. It follows that

$$X_d(e^{j\Omega}) = \frac{1}{T}X_c(j\omega)\big|_{\omega=\frac{\Omega-\pi}{T}}$$

as shown in the following figure.



An alternative solution is to construct a novel "sampling function" that consists of alternating impulses (hence the title of the problem).

$$p_a(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - nT)$$

You can think of this signal as two times an impulse train with period $2T$ minus an impulse train with period T

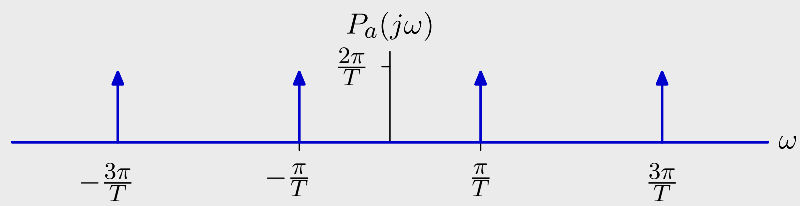
$$p_a(t) = \sum_{n=-\infty}^{\infty} 2\delta(t - 2nT) - \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The Fourier transform is then

$$P_a(j\omega) = \sum_{n=-\infty}^{\infty} 2\,\frac{2\pi}{2T}\,\delta\left(\omega - \frac{2\pi n}{2T}\right) - \sum_{n=-\infty}^{\infty} \frac{2\pi}{T}\,\delta\left(\omega - \frac{2\pi n}{T}\right)$$

$$= \sum_{\substack{n=-\infty \\ n\ \text{odd}}}^{\infty} \frac{2\pi}{T}\,\delta\left(\omega - \frac{\pi n}{T}\right)$$

which is shown below



Convolving this function of frequency with $X_c(j\omega)$ and converting to the DTFT leads to the same solution for $X_d(e^{j\Omega})$ that was shown above.

b. Assume that $x_c(t)$ is bandlimited to $-W \le \omega \le W$. Determine the maximum value of W for which the original signal $x_c(t)$ can be reconstructed from the samples $x_d[n]$.

From the previous part, $\pi - WT$ must be greater than 0. Thus

$$W < \frac{\pi}{T}\,.$$

4. Boxcar sampling

A digital camera focuses light from the environment onto an imaging chip that converts the incident image into a discrete representation composed of pixels. Each pixel represents the total light collected from a region of space

$$x_d[n,m] = \int_{mD-\frac{\Delta}{2}}^{mD+\frac{\Delta}{2}} \int_{nD-\frac{\Delta}{2}}^{nD+\frac{\Delta}{2}} x_c(x,y)\,dx\,dy$$

where Δ is a large fraction of the distance D between pixels. This kind of sampling is often called "boxcar" sampling to distinquish it from the ideal "impulse" sampling that we described in lecture. Assume that boxcar sampling is defined in one dimension as

$$x_d[n] = \int_{nT-\frac{\Delta}{2}}^{nT+\frac{\Delta}{2}} x_c(t)\,dt$$

where T is the intersample "time."

a. Let $X_c(j\omega)$ represent the continuous-time Fourier transform of $x_c(t)$. Determine the discrete-time Fourier transform $X_d(e^{j\Omega})$ of $x_d[n]$ in terms of $X_c(j\omega)$, Δ, and T.

If we define

$$p(t) = \begin{cases} 1 & -\frac{\Delta}{2} < t < \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

then we can express $x_d[n]$ as $x_b(nT)$ where $x_b(t) = (x_c * p)(t)$. The Fourier transform of $x_b(t)$ is then the product of the Fourier transforms of $x_c(t)$ and $p(t)$.

$$P(j\omega) = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^{-j\omega t} dt = \frac{2\sin\frac{\omega\Delta}{2}}{\omega}$$

Provided that $X_b(j\omega)$ is 0 for $|\omega| > \frac{\pi}{T}$,

$$X_d(e^{j\Omega}) = \frac{1}{T} X_b(j\omega)\big|_{\omega=\frac{\Omega}{T}} = \frac{1}{T}\left(X_c(j\omega)P(j\omega)\right)\big|_{\omega=\frac{\Omega}{T}} = \frac{2\sin\frac{\Omega\Delta}{2T}}{\Omega} X_c\left(j\frac{\Omega}{T}\right)$$

over the interval $-\pi < \Omega < \pi$ and repeats periodically (with period 2π) outside that interval.

b. Assume that $x_c(t)$ is bandlimited to $-W \le \omega \le W$. Determine the the maximum value of W for which the original signal $x_c(t)$ can be reconstructed from the samples $x_d[n]$. Compare your answer to the answer for an ideal "impulse" sampler.

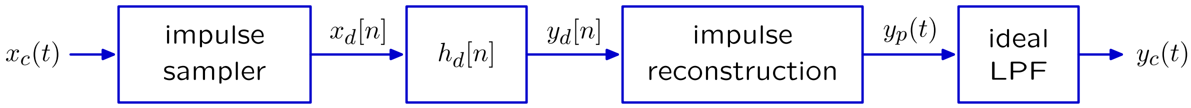
Since $X_b(j\omega)$ must be 0 for $|\omega| > \frac{\pi}{T}$ (see above), it follows that $X_c(j\omega)$ must be zero over this same range. Therefore, X_c must be bandlimited in $W = \frac{\pi}{T}$. Furthermore, $P(j\omega)$ is non-zero for $|\omega| < \pi/T$, so all of the information in X_c can be reconstructed from that in X_b (i.e., there is no loss of information in going from X_c to X_b). Thus boxcar sampling has exactly the same frequency limitations as impulse sampling.

c. Describe the effect of boxcar sampling on the resulting samples $x_d[n]$. How are the samples that result from boxcar sampling different from those that result from impulse sampling?

The samples generated by boxcar sampling are lowpass filtered relative to those generated by impulse sampling. The attenuation is greatest ($\frac{2}{\pi}$) at the maximum frequency $\Omega = \pi$ and for the largest possible value of Δ, which is T. The attenuation is less for lower frequencies and for smaller values of Δ.

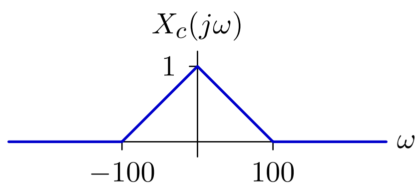
5. DT processing of CT signals

Sampling and reconstruction allow us to process CT signals using digital electronics as shown in the following figure.



The "impulse sampler" and "impulse reconstuction" use sampling interval $T = \pi/100$. The unit-sample function $h_d[n]$ represents the unit-sample response of an ideal DT low-pass filter with gain of 1 for frequencies in the range $-\frac{\pi}{2} < \Omega < \frac{\pi}{2}$. The "ideal LPF" passes frequencies in the range $-100 < \omega < 100$. It also has a gain of T throughout its pass band.

Assume that the Fourier transform of the input $x_c(t)$ is $X(j\omega)$ shown below.



Determine $Y_c(j\omega)$.

Impulse sampling of $x_c(t)$ produces $x_d[n] = x_c(nT)$. Substituting into the transform relations shows that

$$x_d[n] = \frac{1}{2\pi}\int_{2\pi} X_d(e^{j\Omega})e^{j\Omega n}d\Omega = x_c(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\omega)e^{j\omega nT}d\omega$$

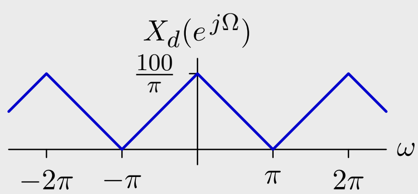
Thus

$$X_d(e^{j\Omega}) = \frac{1}{T}X_c(j\omega)\big|_{\omega=\frac{\Omega}{T}}$$

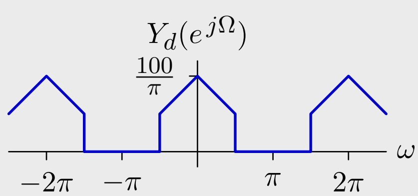
since $d\omega = \frac{1}{T}d\Omega$. Therefore

$$X_d(e^{j\Omega}) = \frac{1}{T}X_c\left(j\frac{\Omega}{T}\right) = \frac{100}{\pi}X_c\left(j\frac{\Omega}{\pi/100}\right) .$$

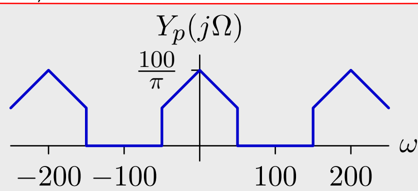
Notice that the cutoff frequency $\omega = 100$ maps to $\Omega = \omega T = 100 \times \frac{\pi}{100} = \pi$. Also, $X_d(e^{j\Omega})$ is periodic in 2π as are all DT Fourier transforms, as shown below.



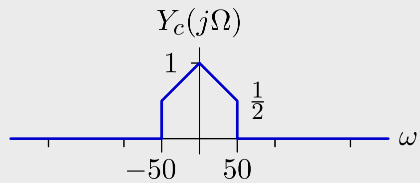
The output of the ideal lowpass filter has the following transform.



Impulse reconstruction then generates a signal with the following transform.



The final output has the following transform, where the DC value is $\frac{100}{\pi} \times T = 1$.



The overall effect is that the input signal is lowpass filtered by the discrete system.