6.003 (Spring 2010)

Quiz #2 *April 7, 2010*

Name:

Kerberos Username:

Please circle your section number:

Section	*Instructor*	*Time*
1	Peter Hagelstein	10 am
2	Peter Hagelstein	11 am
3	Rahul Sarpeshkar	1 pm
4	Rahul Sarpeshkar	2 pm

Grades will be determined by the correctness of your answers (explanations are not required).

Partial credit will be given for ANSWERS that demonstrate some but not all of the important conceptual issues.

You have **two hours.**

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

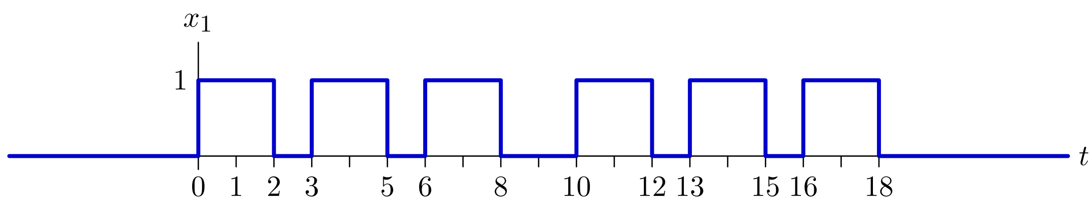
This quiz is closed book, but you may use two 8.5×11 sheets of paper (four sides total).

No calculators, computers, cell phones, music players, or other aids.

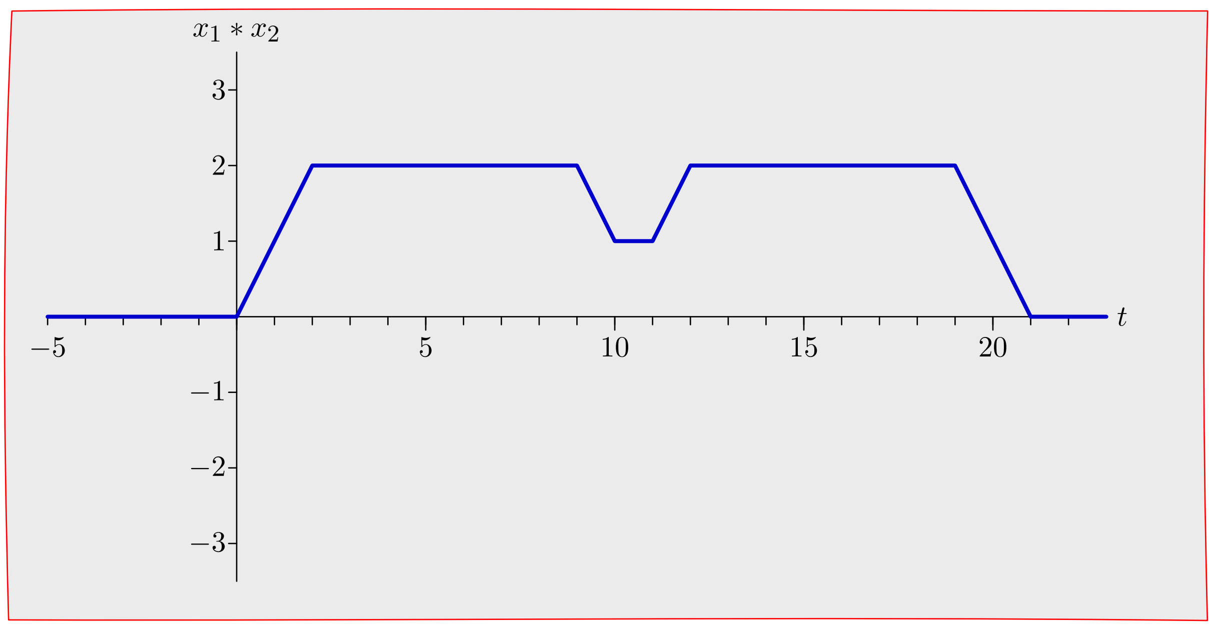
1	/20
2	/30
3	/20
4	/30
Total	/100

1. Convolution *[20 points]*

Signals $x_1(t)$ and $x_2(t)$ are shown in the plots below, and are zero outside the indicated intervals.



Plot the result of convolving $x_1(t)$ with $x_2(t)$.
Make sure that the important break-points are clear.



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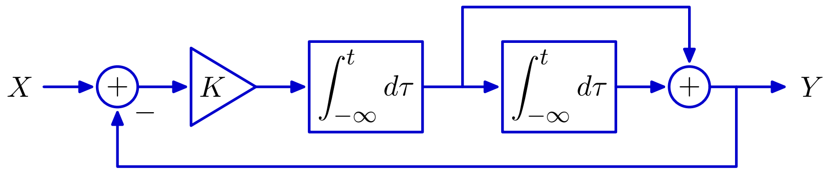
2. Impulse response *[30 points]*

Consider the following control system where the gain K is a real-valued constant.



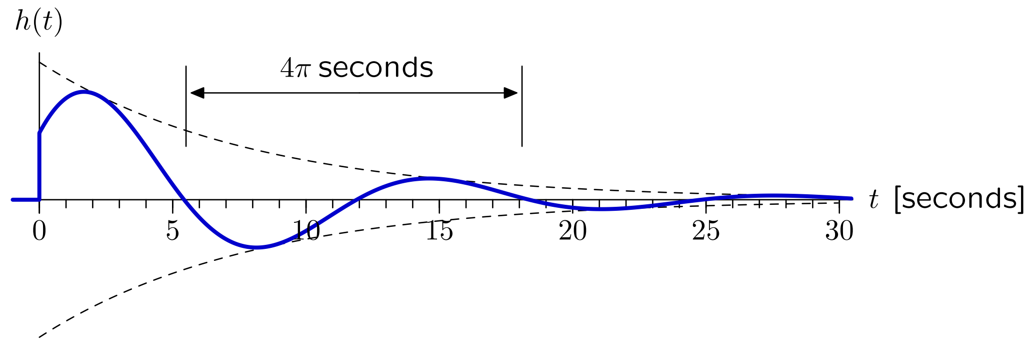
Part a. *[10 points]* Determine the system function $H(s) = \frac{Y(s)}{X(s)}$ as a function of K.

$$H(s) = \boxed{\frac{K(s+1)}{s^2 + Ks + K}}$$

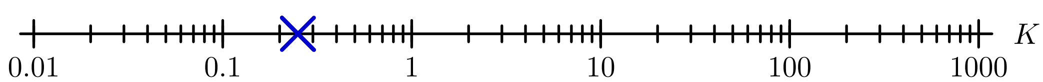
The forward signal path (from the output of the left adder to the output of the right adder) is $K\frac{1}{s} + K\frac{1}{s^2} = K\frac{s+1}{s^2}$. By Black's equation,

$$H(s) = \frac{\frac{K(s+1)}{s^2}}{1 + \frac{K(s+1)}{s^2}} = \frac{K(s+1)}{s^2 + Ks + K}$$

Part b. *[20 points]* The following plot shows the impulse response $h(t)$ of the closed-loop system for a particular value of K. [The dashed curves are exponential functions of time, shown for reference.]



Determine K and indicate its value by placing an X on the following scale.



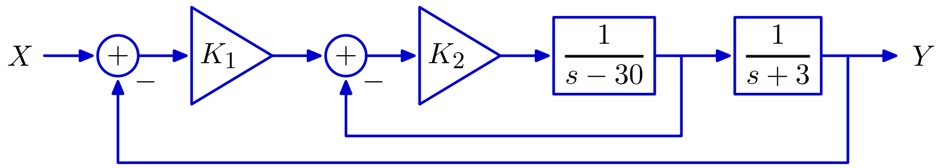
The pole locations can also be found from the impulse response:

$$s = \frac{1}{\tau} \pm j\omega_d \approx \frac{1}{8} \pm j\frac{2\pi}{4\pi} \approx \frac{1}{8} \pm j\frac{1}{2}$$

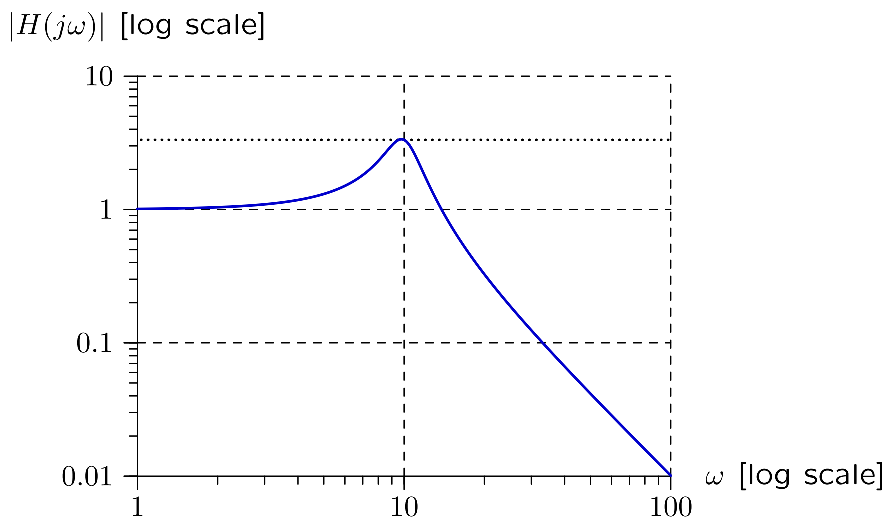
where τ is the time constant of the exponential decay (approximately 8 seconds) and the frequency ω_d of the oscillation is 2π over the period of the oscillation (4π seconds). Then $(s + \frac{1}{8} + j\frac{1}{2})(s + \frac{1}{8} - j\frac{1}{2}) = s^2 + \frac{1}{4}s + \frac{1}{64} + \frac{1}{4}$. Thus $K \approx \frac{1}{4}$.

3. Frequency Response *[20 points]*

Let $H(s) = Y(s)/X(s)$ for the following system.



Find K_1 and K_2 so that $|H(j\omega)|$ matches the plot below.



Enter numbers (or numerical expressions) for K_1 and K_2 in the boxes.

$K_1 =$ $\frac{10}{3} \approx 3.33$

$K_2 =$ 30

The system function for the inner loop is

$$\frac{\frac{K_2}{s-30}}{1+\frac{K_2}{s-30}} = \frac{K_2}{s+K_2-30}$$

Thus the overall system function is

$$H(s) = \frac{K_1\left(\frac{K_2}{s+K_2-30}\right)\left(\frac{1}{s+3}\right)}{1+K_1\left(\frac{K_2}{s+K_2-30}\right)\left(\frac{1}{s+3}\right)} = \frac{K_1K_2}{s^2+(K_2-27)s+3K_2-90+K_1K_2}$$

At low frequencies, the magnitude of the system function must be 1:

$$|H(j0)| = \frac{K_1K_2}{3K_2-90+K_1K_2} = 1\,.$$

At $\omega = 100$ (i.e., a high frequency), the magnitude of the system function must be 10^{-2}:

$$|H(j100)| = \frac{K_1 K_2}{\omega^2} = \frac{K_1 K_2}{100^2} = 10^{-2}\,.$$

The latter implies that $K_1 K_2 = 100$, so that the former yields $\frac{100}{3K_2 - 90 + 100} = 1$, so that $K_2 = 30$ and $K_1 = 100/30 = 10/3$.

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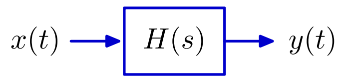
4. Fourier series *[30 points]*

Let $x(t)$ represent a periodic signal (period $T = 8$ seconds) whose Fourier series coefficients are

$$a_k = \begin{cases} \frac{1}{j\pi k} & \text{for integers } k \neq 0 \\ 0 & k = 0. \end{cases}$$

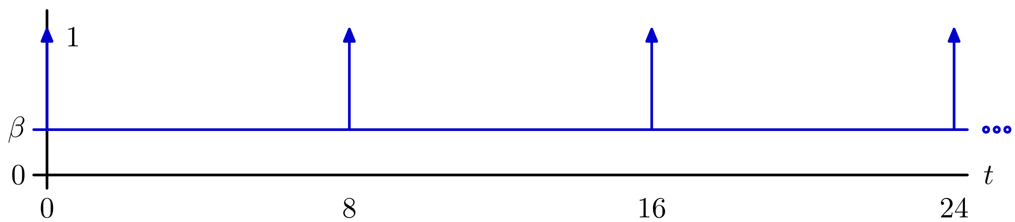
When $x(t)$ is the input to a linear, time-invariant system with system function $H(s)$



the output $y(t)$ is the sum of a constant β plus a periodic train of impulses with area 1,

$$y(t) = \beta + \sum_{k=-\infty}^{\infty} \delta(t - 8k)$$

as shown below.



Part a. *[15 points]* Determine β.

$\beta =$ $-\frac{1}{8}$

The Fourier series coefficients of $y(t)$ are

$$b_k = \frac{1}{8}\int_8 (\delta(t) + \beta)e^{-j\frac{2\pi}{8}kt}dt = \begin{cases} \frac{1}{8} + \beta & k = 0 \\ \frac{1}{8} & \text{k>0} \end{cases}$$

However b_0 must be zero, since $a_0 = 0$. Therefore, $\beta = -\frac{1}{8}$.

Part b. *[15 points]* Consider the output of the same system if the period of the input signal is changed to $T = 4$ seconds while keeping the Fourier series coefficients a_k unchanged.

Is it possible to determine the new output signal from the information provided?

possible? (**Yes** or **No**) Yes

From part a, we know the frequency response of the system at frequencies that are integer multiples of $\frac{2\pi}{8}$ radians/second:

k	ω	$H(j\omega) = \frac{b_k}{a_k} = \frac{j\pi k}{8}$
1	$2\pi/8$	$j\pi/8$
2	$4\pi/8$	$j2\pi/8$
3	$6\pi/8$	$j3\pi/8$
4	$8\pi/8$	$j4\pi/8$
...		

For part b, we need to know the frequency response of the system at frequencies that are integer multiples of $\frac{2\pi}{4}$ radians/second, which is just every other line in the table above:

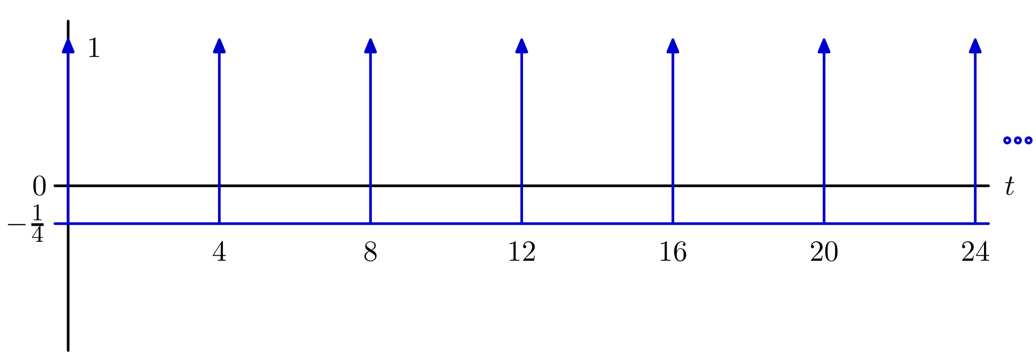
k	ω	$H(j\omega)$
1	$2\pi/4$	$j2\pi/8$
2	$4\pi/4$	$j4\pi/8$
3	$6\pi/4$	$j6\pi/8$
4	$8\pi/4$	$j8\pi/8$
...		

Thus, each harmonic in the new signal is scaled by twice the previous scale factor.

The constant is therefore $-\frac{1}{4}$ (twice $-\frac{1}{8}$).

The areas of the impulses in part a were 1. Thus the magnitudes of the corresponding harmonics were $\frac{1}{8}$. Doubling these magnitudes gives harmonics with magnitudes of $\frac{1}{4}$, so the areas of the impulses are still 1.

If **Yes**, sketch and fully label the new output signal on the axes below.



If **No**, briefly explain why not.

Worksheet (intentionally blank)

Worksheet (intentionally blank)