

Magnetostatics (Continued)

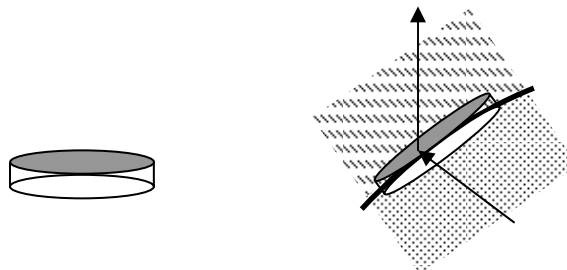
Summary from last time:

<p>Differential Form</p> $\nabla \cdot \vec{B} = 0$ $\nabla \times \vec{B} = \mu_0 \vec{J}$ $\nabla \cdot \vec{A} = 0$ $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ $\vec{B} = \nabla \times \vec{A}$
<p>Units</p> <p>B Tesla</p> <p>A Tesla-m</p> <p>μ_0 Tesla-m/Ampere = Henries/m</p>

<p>Integral Form</p> $\oiint \vec{B} \cdot d\vec{A} = 0$ <p><i>any closed surface</i></p> $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \iint \vec{J} \cdot d\vec{A}$ <p><i>closed loop</i> <i>surface defined by loop</i></p> $\vec{B} = \frac{\mu_0}{4\pi} \iiint_{\text{volume containing } \vec{J}} \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{ \vec{r} - \vec{r}' ^3} dV'$ $\vec{A} = \frac{\mu_0}{4\pi} \iiint_{\text{volume containing } \vec{J}} \frac{\vec{J}(\vec{r}')}{ \vec{r} - \vec{r}' } dV'$
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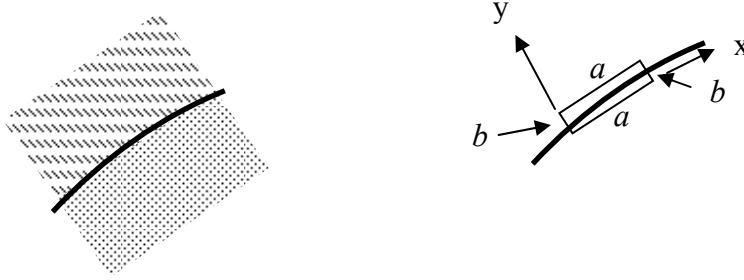
Boundary Conditions

Similar to electrostatics, magnetostatic problems often involve finding the magnetic field in a configuration involving prescribed currents in the presence of conducting bodies. To solve such problems, it is necessary to know how \vec{B} behaves across a boundary, for example a conducting-vacuum interface. The situation for the normal field component is similar to that in electrostatics, except that the RHS of Gauss' Law is rigorously 0.



Considering the "Gaussian pillbox" in the left of the figure mounted as sketched on an interface as shown on the right, it is easily argued that "Normal \vec{B} must be Continuous".

Next consider the boundary and Stokesian contour sketched below. Applying Ampère's Law in



integral form to the contour, we get $B_x^- a + B_y(x=a)b - B_x^+ a - B_y(x=0)b = \mu_0 J_z ab$. Now let the height of the box b shrink essentially to 0. One would be tempted to conclude that the tangential component of \vec{B} is continuous. This is often true, *except* in the case of ideal conductors ($\sigma = \infty$) where a surface current may exist. Then $J_z b \rightarrow K_z$, where K_z is the current per unit length (in the z -direction) along the surface. The result of applying Ampère's Law to the contour is then $B_x^- - B_x^+ = \mu_0 K_z$. Rotating the contour so that it lies in the $x=0$ plane yields

$B_z^- - B_z^+ = -\mu_0 K_x$. In vector form, $\tan(\vec{B}^- - \vec{B}^+) = \mu_0 \hat{y} \times \vec{K}$ or more generally,

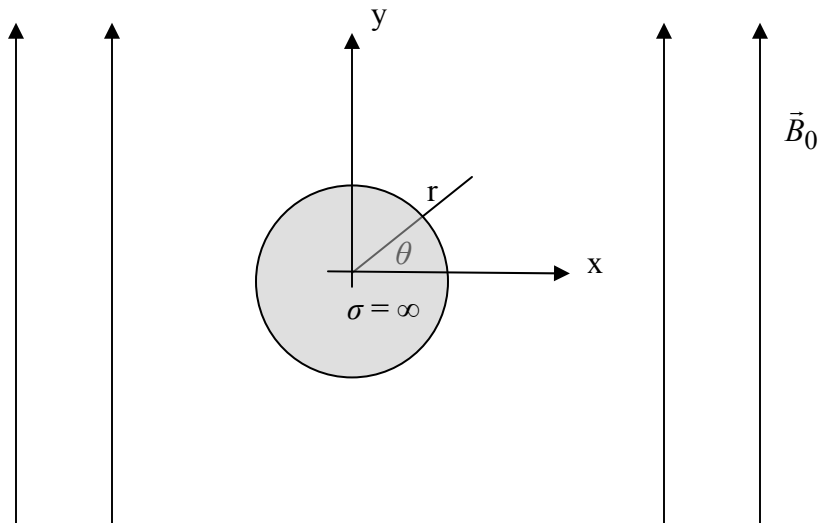
$$\vec{B}^- - \vec{B}^+ = \mu_0 \hat{n} \times \vec{K} \quad (1)$$

where \hat{n} is the unit normal to the interface, pointing from the - surface to the + one. Notice that Eq(1) takes care of the normal \vec{B} boundary condition as well. We can also use Eq(1) to determine the surface current by crossing it with \hat{n} to get

$$\mu_0 \vec{K} = \hat{n} \times (\vec{B}^+ - \vec{B}^-)$$

Application of this boundary condition will be clarified with the aid of an example.

Example: Perfectly conducting cylinder in uniform applied B field.



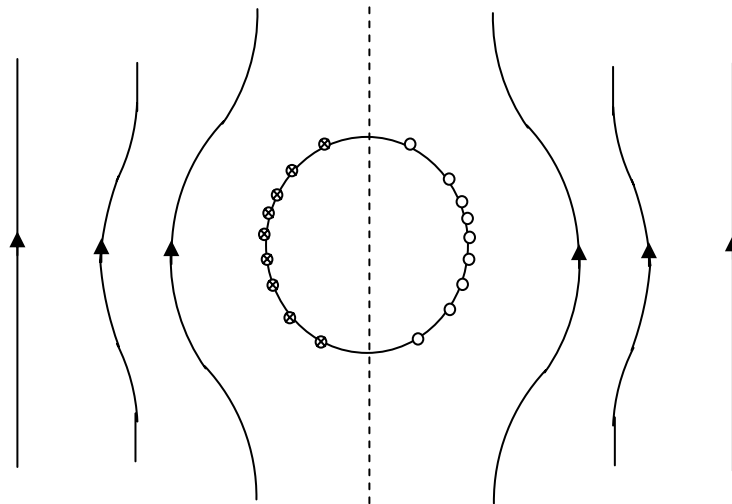
The figure suggests a perfectly conducting cylinder immersed in a uniform applied magnetic field, $\vec{B}_{app} = \hat{z}B_0 = (\hat{r} \sin \theta + \hat{\theta} \cos \theta)B_0$ where r, θ are the usual cylindrical coordinates. Since the cylinder is a perfect conductor, $\vec{B} = 0$ everywhere inside it. (This actually follows from Faraday's Law, which we'll get to next week.) Finding \vec{B} outside the cylinder requires application of the boundary conditions, the most important of which is $B_r = 0$ on the cylinder surface. The structure of the applied field suggests that we try a solution composed of the applied field plus a that of a cylindrical dipole (See lecture 2, page 6.):

$$\vec{B} = A \frac{R^2}{r^2} (\hat{r} \sin \theta - \hat{\theta} \cos \theta) + B_0 (\hat{r} \sin \theta + \hat{\theta} \cos \theta)$$

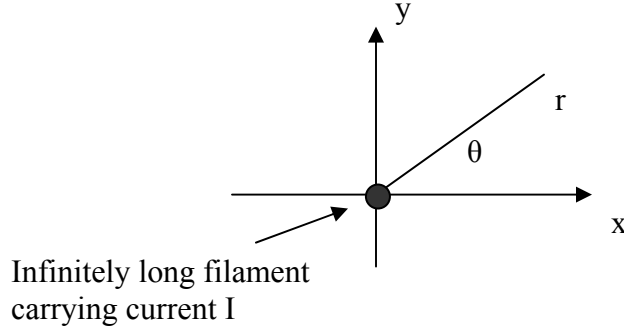
where R is the radius of the cylinder. Since $B_r = 0 @ r = R$, $A = -B_0$. What about $\tan \vec{B}$? $\tan \vec{B}$ is in fact discontinuous, i.e., $B_\theta = 2B_0 \cos \theta$ just outside the cylinder and is 0 just inside. This is possible because a surface current flows on the cylinder; it's in the z -direction and given by

$$\vec{K} = \hat{z} \frac{2}{\mu_0} B_0 \cos \theta.$$

A schematic indication of the current flowing on the surface of the cylinder, and a few field lines in the plane $z = const$ is shown in the sketch below.



2-D Magnetics



Start with the equations for \vec{B} and \vec{A} for a *long*, straight current filament. Referring to the figure on the top of the page, observe that $\frac{\partial}{\partial z} = \frac{\partial}{\partial \theta} = 0$. Since $\vec{B} = \hat{\theta} B_\theta$ we find, by applying Ampère's

Law to a circle with center on the filament and enclosing it, $\vec{B} = \hat{\theta} \frac{\mu_0 I}{2\pi r}$. Also, since

$$\vec{A} = \hat{z} A(r) \text{ and } \vec{B} = \nabla \times \vec{A} = -\hat{\theta} \frac{\partial A}{\partial r} \text{ we find } \vec{A} = \hat{z} \frac{\mu_0 I}{2\pi} \ln \frac{1}{r}.$$

If we move the filament to a point \vec{r}' , the vector potential becomes $\vec{A} = \hat{z} \frac{\mu_0 I}{2\pi} \ln \frac{1}{|\vec{r} - \vec{r}'|}$. Finally, if the current is distributed in the x - y plane, but still infinite in the z -direction, $I \rightarrow J_z dS$ and the vector potential becomes:

$$\vec{A} = \hat{z} \frac{\mu_0}{2\pi} \iint_S \ln \frac{1}{|\vec{r} - \vec{r}'|} J_z(\vec{r}') dS'. \quad (2)$$

where the surface S is in the plane $z = \text{constant}$ and extends over the entire 2D region where there is current. This is the 2D analogue to the 3D version given on page 1. Notice the similarity of this result to the solution of 2D potential problems:

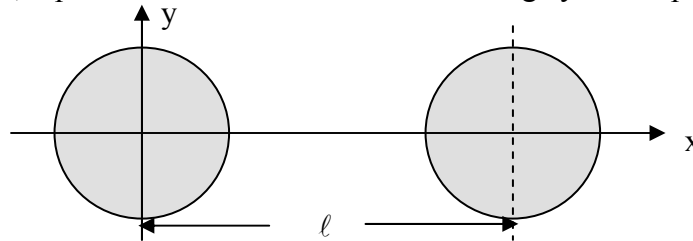
$$\phi = \frac{1}{2\pi\epsilon_0} \iint_S \ln \frac{1}{|\vec{r} - \vec{r}'|} \rho(\vec{r}') dS'$$

For magnetostatic problems with perfect conductors, A_z is constant on all conductors, just as ϕ would be for the corresponding electrostatic problem. Therefore we have the correspondence

$$\phi \Leftrightarrow A_z.$$

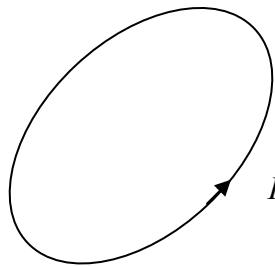
The point is that any problem involving conductors and charge in an electrostatic problem now becomes a candidate for a solution of an analogous magnetostatic problem. For example,

consider the electrostatic problem posed in Lecture 4, bottom of pg 6. The figure, reproduced below for convenience, represents the cross-section of two long cylinders placed a distance l



apart. The left (right) cylinder has potential $V/2$ ($-V/2$) and the radius of each cylinder is R . As was pointed out in the discussion of this problem in Lecture 4, the potential can be found by replacing the cylinders by line charges, one of strength λ , located at $x = a$, the other of strength $-\lambda$ and located at $x = l-a$. The condition for this to yield the solution is $a(l-a) = R^2$. The application of this approach to finding the vector potential for a problem where the cylinders are perfectly conducting and carry the currents $\pm I$ should be obvious.

Flux and Inductance



Consider the loop of current sketched above. The flux Φ through the loop is defined as

$$\Phi \equiv \iint_{\substack{\text{Surface} \\ \text{bounded} \\ \text{by loop}}} \vec{B} \cdot \hat{n} dS \quad \text{Webers (or Volt - sec)}$$

Any surface can be chosen as long as its perimeter conforms to the current loop; then, Gauss' Law for \vec{B} guarantees that Φ will be uniquely defined. By convention, one chooses the direction of \hat{n} according to the right hand rule. With thumb aligned along the direction of I , the fingers point in the direction of \hat{n} .

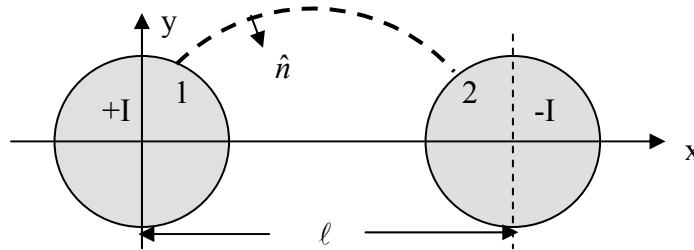
The inductance of the loop L is defined as the flux per unit current:

$$L = \frac{\Phi}{I} \text{ Henries}$$

Notice that the unit for \vec{B} is Webers/m². Since $\vec{B} = \nabla \times \vec{A}$, Stokes' Theorem tells us immediately that Φ and L can be written in terms of \vec{A} :

$$\Phi = \oint \vec{A} \cdot d\vec{l} \quad L = \frac{\oint \vec{A} \cdot d\vec{l}}{I}$$

In 2D problems, the flux and inductance are defined per unit length parallel to the direction of



current. In the case shown above,

$$\Phi = \int_1^2 \vec{B} \cdot \hat{n} dl = A_z(1) - A_z(2) \quad W, \quad L = \frac{\Phi}{I} \quad H/m.$$

Magnetic Materials

Atoms in a material often possess a magnetic dipole moment, and the dipoles can be either partially aligned even without an applied field or partially aligned due to the presence of a field. As we've seen, an applied \vec{B} -field tends to align dipoles through the action of a torque, $\vec{m} \times \vec{B}$. The alignment of the dipoles in turn modifies the field and we need to develop macroscopic equations accounting for this effect.

The vector potential from an elementary dipole is $\vec{A}_{dp} = \hat{\phi} \frac{\mu_0 m \sin \theta}{4\pi r^2}$, as shown at the end of Lecture 6. For a current ring, $m = \pi a^2 I$. The term $\hat{\phi} m \sin \theta$ can be written in vector form as $\vec{m} \times \hat{r}$, where $\vec{m} = \hat{z} m$. This allows us to write down an expression for \vec{A}_{dp} that is valid for any orientation:

$$\vec{A}_{dp} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3},$$

and this expression also allows us to move the dipole to any point in space and calculate \vec{A}_{dp} :

$$\vec{A}_{dp} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.$$

Now suppose $\vec{M}(\vec{r})$ is the net dipole moment per unit volume ($\frac{1}{\Delta Vol} \sum \vec{m}_i$), then the total contribution to \vec{A} is

$$\begin{aligned}
\vec{A}_M &= \frac{\mu_0}{4\pi} \int \vec{M}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d\vec{r}' \\
&= \frac{\mu_0}{4\pi} \int \vec{M}(\vec{r}') \times \nabla_{r'} \frac{1}{|\vec{r} - \vec{r}'|} d\vec{r}' \\
&= -\frac{\mu_0}{4\pi} \int \nabla_{r'} \times \left(\frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) d\vec{r}' + \frac{\mu_0}{4\pi} \int \frac{\nabla_{r'} \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}'
\end{aligned}$$

The first term in the last expression integrates to 0, assuming that $\vec{M}(\vec{r})$ vanishes for large r , i.e., that the magnetic material is finite.

If now we add in the effect of currents (the last equation on page 1 under “integral equations”, we get:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') + \nabla_{r'} \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}' \quad (3)$$

This equation shows that the dipoles in a magnetic material produce exactly the same effect as a current density given by $\vec{J}_{eff} = \nabla \times \vec{M}(r)$. It is easy to see that $\nabla \cdot \vec{A}$ is still 0, and therefore:

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \nabla \times \vec{M} .$$

So the differential equations for \vec{B} are:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \nabla \times \vec{M},$$

while it is still the case that $\vec{B} = \nabla \times \vec{A}$ with \vec{A} computed from (3). In fact, all the equations on page 1 remain valid with $\vec{J} \rightarrow \vec{J} + \nabla \times \vec{M}$.

It remains to know something about \vec{M} . The standard approach is to define a new vector, \vec{H} , the so-called magnetic intensity vector which is defined by the expression

$$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}$$

The magnetization vector \vec{M} is then defined in terms of its relationship to \vec{H} , i.e., $\vec{M} = \vec{M}(\vec{H})$. In terms of \vec{H} our equations become

$$\begin{aligned}\nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{H} &= \vec{J} \\ \vec{B} &= \mu_0(\vec{H} + \vec{M}) \\ \vec{M} &= \vec{M}(\vec{H}),\end{aligned}$$

where the latter relation comes from the properties of the material. These are the differential equations appropriate for treating magnetostatic problems with magnetic material.

For linear isotropic materials, $\vec{M} = \chi_m \vec{H}$ so

$$\vec{B} = \mu_0(1 + \chi_m)\vec{H} \equiv \mu\vec{H}$$

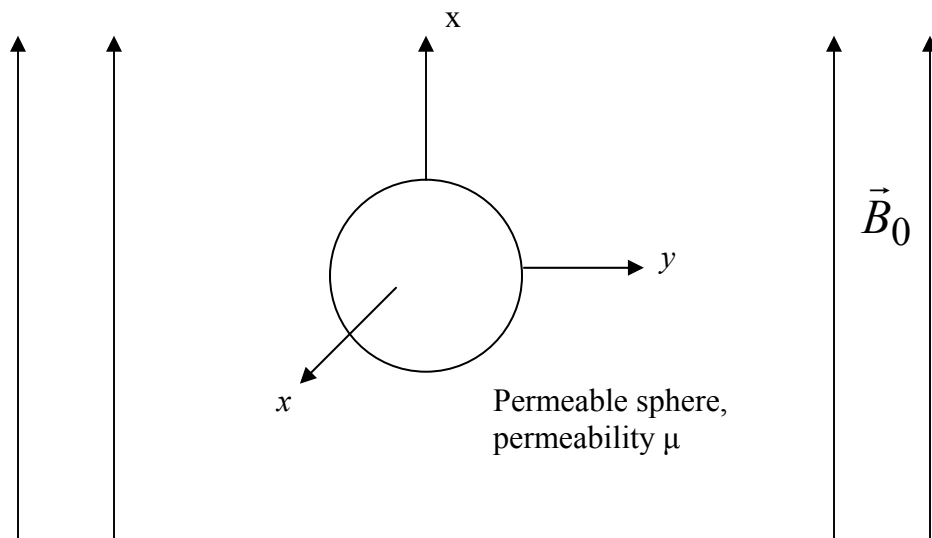
where $\mu = \mu_0(1 + \chi_m)$. The quantity χ_m is the “magnetic susceptibility” while μ is known as the permeability. The boundary conditions are then

$$\vec{B}^- \cdot \hat{n} = \vec{B}^+ \cdot \hat{n}$$

$$\tan \vec{H}^- - \tan \vec{H}^+ = \hat{n} \times \vec{K},$$

the latter equation coming from the arguments that led to (1).

Example: A permeable sphere immersed in a uniform field



The figure shows a permeable sphere, radius R , immersed in a uniform magnetic field $\vec{B}_{app} = \hat{z}B_0$. The region outside the sphere is vacuum, $\mu = \mu_0$.

Start by writing $\vec{B}_{app} = (\hat{r} \cos \theta - \hat{\theta} \sin \theta)B_0$, which is suggestive of using the following solutions for \vec{B} :

$$\text{For } r > R, \quad \vec{B} = \vec{B}_{app} + \vec{B}_{dp} = B_0(\hat{r} \cos \theta - \hat{\theta} \sin \theta) + B_{dp} \frac{R^3}{r^3}(2\hat{r} \cos \theta + \hat{\theta} \sin \theta);$$

$$\text{For } r < R, \quad \vec{B} = B_{in}(\hat{r} \cos \theta - \hat{\theta} \sin \theta)$$

Boundary conditions:

$$\text{Normal } \vec{B} \text{ continuous: } B_0 + 2B_{dp} = B_{in}$$

$$\text{Tangential } \vec{H} \text{ continuous: } \frac{-B_0 + B_{dp}}{\mu_0} = \frac{-B_{in}}{\mu}$$

Solution:

$$B_{dp} = \frac{\mu_r - 1}{\mu_r + 2} B_0, \quad B_{in} = \frac{3\mu_r}{\mu_r + 2} B_0, \quad \text{where } \mu_r = \frac{\mu}{\mu_0}.$$

For large μ_r , $B_{dp} = B_0$, $B_{in} = 3B_0$. Suppose a very small spherical hole with radius a is cut into the sphere. What is \vec{B} in the hole?

$$\text{For } r > a, \quad \vec{B} = 3\vec{B}_{app} + \vec{B}'_{dp} = 3B_0(\hat{r} \cos \theta - \hat{\theta} \sin \theta) + B'_{dp} \frac{a^3}{r^3}(2\hat{r} \cos \theta + \hat{\theta} \sin \theta)$$

$$\text{For } r < a, \quad \vec{B} = B'_{in}(\hat{r} \cos \theta - \hat{\theta} \sin \theta)$$

Applying boundary conditions in the same way as above, we find $B'_{in} = \frac{9}{1 + \mu_r} B_0 \ll B_0$. Since

$\mu_r \rightarrow 10^3 - 10^4$ for iron, for example, this illustrates that magnetic materials can be very effective in shielding out applied magnetic fields!