

# Chapter 6

## Analogy

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When the going gets tough, the tough lower their standards. It is the creed of the sloppy, the lazy, and any who want results. This chapter introduces a technique, reasoning by **analogy**, that embodies this maxim. It works well with extreme-case reasoning.

### 6.1 Tetrahedral bond angle

What is the bond angle in methane,  $\text{CH}_4$ ? The carbon sits at the centroid of a regular tetrahedron, and the hydrogens sit at the vertices. Trigonometry and analytic geometry solve this problem, but let's try analogy. Three dimensions is hard to visualize and figure out, so lower your standards: Look for a simpler problem that preserves its essentials. What is essential is not always obvious, and you might solve several simpler variants before discovering those features.

Let's try the simplest change, going from three dimensions to two dimensions. The two-dimensional version of the problem is to find the bond angle in a planar molecule, for example  $\text{NH}_3$  smashed into a plane. The bond angle is one-third of a full circle or  $120^\circ$ . The center of the bond angle is the centroid is the centroid of the object, so its location might be relevant in solving the problem. Who knows where a tetrahedron's centroid is; but the triangle has a centroid one-third of the way from one edge to the opposite vertex.

Here is a table with this data, where

$d$  is the number of dimensions.

It's hard to generalize from such sparse data, reflected by the question marks in the tetrahedron row.

Here is where extreme-cases reasoning helps. You can get **free**

**data** by extending the analogy to a yet more extreme problem. If two dimensions are easier than three, then one dimension should be easier than two.

|             | $d$ | centroid | $\theta$ |
|-------------|-----|----------|----------|
| triangle    | 2   | 1/3      | 120°     |
| tetrahedron | 3   | ?        | ?        |

In one dimension, the object is a

line. The centroid is one-half of the way between the endpoints.

The bond angle is 180°. And the table now is more complete. The

bond angle has several generalizations to  $d = 3$ , depending on

what pattern underlies it. For

example, if the pattern is  $\theta = (240 - 60d)^\circ$ , then  $\theta(d = 3) = 60^\circ$ . Having

made a conjecture, it is important to test your conjecture. To conjecture

and not to test – the great mathematician and mathematics teacher George

Polya [24] says that to do so is the mark of a savage! So: Is that conjecture

reasonable? It's dubious because, first, the angle is less than 90°.

If the molecule were  $\text{CH}_6$ , with the carbon at the center of a cube and the hydrogens

at the faces of a cube, then the bond angle would be exactly 90°.

With only four hydrogens, rather than six, the bond angle should be larger than 90°.

So 60° seems to be a dubious conjecture. For a second reason that it is dubious,

the try a more extreme case: four dimensions. Then, according to the

$(240 - 60d)^\circ$  conjecture, the bond angle would be zero, which is nonsense.

So the conjecture is dubious on several grounds.

Let's make another conjecture. What about  $\theta = 360^\circ / (d + 1)$ ? That conjecture

fits  $d = 1$  and  $d = 2$ . For  $d = 3$  it predicts  $\theta = 90^\circ$ . By the reasoning

that rejected the previous conjecture, this angle is too small. Furthermore, it

means that for  $d = 4$ , the angle drops below 90°. That's also not reasonable.

| <i>shape</i> | $d$ | centroid | $\theta$ |
|--------------|-----|----------|----------|
| line         | 1   | 1/2      | 180°     |
| triangle     | 2   | 1/3      | 120°     |
| tetrahedron  | 3   | ?        | ?        |

## 6.1 Tetrahedral bond angle

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To help find another conjecture, it's time for a new idea. Instead of guessing the bond angle directly, guess a function of it that makes it easier to guess. The bond angle, if we solve it honestly,

| <i>shape</i> | <i>d</i> | centroid | $\theta$ | $\cos \theta$ |
|--------------|----------|----------|----------|---------------|
| line         | 1        | 1/2      | 180°     | -1            |
| triangle     | 2        | 1/3      | 120°     | -1/2          |
| tetrahedron  | 3        | ?        | ?        | ?             |

will come from the dot product of two vectors (the vectors from a vertex to the centroid of the opposite face). Dot products produce cosines, so perhaps  $\cos \theta$  is easier to guess than  $\theta$  itself. This idea adds a column to the table.

One possible pattern for  $\cos \theta$  is  $-2^{1-d}$ , which fits the  $d = 1$  and  $d = 2$  data. For  $d = 3$  it predicts  $\cos \theta = -1/4$ , which means  $\theta > 90^\circ$ , an excellent result. In the extreme case of  $d \rightarrow \infty$  it predicts that  $\theta \rightarrow 90^\circ$ . Let's check that result. The  $d$ -dimensional problem has a carbon at the center and  $d + 1$  hydrogens at the vertices of the object. That bond angle should be more than  $90^\circ$ : The problem with  $90^\circ$  bonds has  $2d$  hydrogens, each at center of the  $2d$  faces of a  $d$ -dimensional cube. And  $d + 1$  hydrogens should be more spread out than  $2d$  hydrogens. So the  $-2^{1-d}$  is not reasonable, although it got off to a good start.

To find another conjecture, look at the pattern in the centroid column. It is  $1/(d + 1)$ . So  $1/(d + 1)$  or  $1/d$  might be a reasonable fit for  $\cos \theta$ . Perhaps  $\cos \theta = -1/d$ ? That fits the  $d = 1$  and  $d = 2$  data, and predicts  $\cos \theta = -1/3$  and  $\theta \approx 109.47^\circ$ . The only problem is that this conjecture also predicts that  $\theta \rightarrow 90^\circ$  as  $d \rightarrow \infty$ . So maybe that's okay?

Anyway, the more likely conjecture, because it respects the pattern in the centroid column, is that  $\cos \theta = -1/d$ . Let's see if we can check that. Yes! But first see if we can check the centroid conjecture, since the  $\cos \theta$  one depends on it. And we can check that too. It says that the height is  $1/(d + 1)$  of the way from the base. Hmm,  $d + 1$  – that's how many hydrogens there are. And 1, the numerator, is how many hydrogens are not on the base. Indeed, the average height of the  $d + 1$  vertices is  $1/(d + 1)$  – which explains the centroid location.

Now, knowing where the centroid is, look at a cross-section of the tetrahedron. The cosine of the complement of  $\theta$  is

$$\cos(180^\circ - \theta) = \frac{1/(d + 1)}{d/(d + 1)} = \frac{1}{d}.$$

Since  $\cos \theta = -\cos(180^\circ - \theta)$ , the result is

$$\cos \theta = -\frac{1}{d}.$$

The final verifications are elegant arguments, ones that we might not have thought of on first try. That's okay. Here's what friends who went to the US Math Olympiad training session told me they were taught: Find the answer by any cheap method that you can find; once you know, or are reasonably sure of the answer, you often can then find a more elegant method and never mention the original cheap methods.

I agree with that philosophy, except for one point. It is worthwhile mentioning the cheap methods, because, just as they were useful in this problem, they will be useful in other problems.

## 6.2 Steiner's plane problem

A famous problem is Steiner's plane problem: Into how many regions do five planes divide space? There are lots of answers to this question, some boring. If the planes are parallel, for example, they make six regions. If the planes are not parallel, the number grows. But the number of regions depends on how 'unparallel' the planes are. So assume that the planes are in a random orientation, to remove the chance of a potential region being wiped out by a silly coincidence.

Five planes are hard to imagine and hard to build. An analogous problem is the same question with four planes. That's still hard, however. So try three planes. That's also hard, so try two planes. That's easy: four regions. Don't forget the more extreme case of one plane: two regions. And more free data comes from the most extreme case of zero planes: one region. So, starting with  $n = 0$  planes, the number of regions is: 1, 2, 4, . . . . Are those powers of two, and is the next number in the sequence 8? Start with two planes making four regions. Place the third plane to cut the other two, so that it splits each region into two pieces – making eight regions total. So 8 is indeed the next number. Is 16 and then 32 next? That is represented in the following table:

|   |   |   |   |   |     |     |
|---|---|---|---|---|-----|-----|
| n | 0 | 1 | 2 | 3 | 4   | 5   |
| r | 1 | 2 | 4 | 8 | 16? | 32? |

So we have a conjecture, an educated guess, for  $n = 5$ . Its conjectural nature is reflected in the question marks. But how to test it? We still cannot easily visualize four planes, let alone five planes.

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Analogy comes to the rescue again. If fewer planes were easier to solve than more planes, fewer dimensions might also help. So let's study the same problem in two dimensions. What is the analogous problem that preserves the essentials? It cannot be placing  $n$  planes in a plane. Rather, we should also reduce the dimensionality of the placed object: Place  $n$  lines in a plane, in random orientations and positions. How many planar regions does that make? Having learnt the lesson of free data, start with  $n = 0$  lines giving 1 region. One line makes two regions; two lines makes four regions. It looks like powers of two again.

Let's test it with three lines. Here's a picture. They make *seven* regions, not eight. So the conjecture fails. Let's do four lines and count carefully. That's 11 regions, remote from the next power of two, which would have been 16. Here are the results for the two-dimensional region:

|   |   |   |   |   |    |   |
|---|---|---|---|---|----|---|
| n | 0 | 1 | 2 | 3 | 4  | 5 |
| r | 1 | 2 | 4 | 7 | 11 | ? |

Let's combine the two- and three-dimensional data:

|       |   |   |   |   |     |     |
|-------|---|---|---|---|-----|-----|
| n     | 0 | 1 | 2 | 3 | 4   | 5   |
| $r_2$ | 1 | 2 | 4 | 7 | 11  | ?   |
| $r_3$ | 1 | 2 | 4 | 8 | 16? | 32? |

Now once again, use extreme cases and get free data. With data for two and three dimensions, why not include data for one dimension?! In one dimension the problem is, after putting  $n$  points on a line, how many regions (line segments) do they make? That's a fencpost problem, so be careful not to be off by one. When  $n = 0$ , there's only one segment – the whole infinite line. Each dot divides one segment into two, so it increases  $r$  by one. So there will be  $r = n + 1$  regions.

|       |   |   |   |   |     |     |         |
|-------|---|---|---|---|-----|-----|---------|
| n     | 0 | 1 | 2 | 3 | 4   | 5   | n       |
| $r_1$ | 1 | 2 | 3 | 4 | 5   | 6   | $n + 1$ |
| $r_2$ | 1 | 2 | 4 | 7 | 11  | ?   |         |
| $r_3$ | 1 | 2 | 4 | 8 | 16? | 32? |         |

Now we have lots of data! Can you spot a pattern? Look at the connected entries, where  $4 + 7 = 11$ :

|                |   |   |   |   |     |     |       |
|----------------|---|---|---|---|-----|-----|-------|
| n              | 0 | 1 | 2 | 3 | 4   | 5   | n     |
| r <sub>1</sub> | 1 | 2 | 3 | 4 | 5   | 6   | n + 1 |
| r <sub>2</sub> | 1 | 2 | 4 | 7 | 11  | ?   |       |
| r <sub>3</sub> | 1 | 2 | 4 | 8 | 16? | 32? |       |

That pattern holds wherever there is data to check it against. For example,  $3 + 4 = 7$ . Or  $4 + 4 = 8$ . If that's true, then in two dimensions when  $n = 5$ , then  $r = 16$ . In three dimensions, when  $n = 4$ , there are  $r = 15$  regions (one less than the prediction of  $r = 2^n$ ). And with five planes, there will be 26 regions. So, that's our conjecture, which now has lots going for it. Let's now be even more extreme and get one more row of free data: 0 dimensions. In 0 dimensions, the object is a point, and there's only one point no matter how many -1-dimensional objects subdivide it. So  $r = 1$  always. Then:

|                |   |   |   |   |     |     |       |
|----------------|---|---|---|---|-----|-----|-------|
| n              | 0 | 1 | 2 | 3 | 4   | 5   | n     |
| r <sub>0</sub> | 1 | 1 | 1 | 1 | 1   | 1   | 1     |
| r <sub>1</sub> | 1 | 2 | 3 | 4 | 5   | 6   | n + 1 |
| r <sub>2</sub> | 1 | 2 | 4 | 7 | 11  | ?   |       |
| r <sub>3</sub> | 1 | 2 | 4 | 8 | 16? | 32? |       |

And the new row, for 0 dimensions, continues the pattern.

For fun let's fit polynomials to the data we have – before making the conjectured leap. The zeroth row is fit by  $r = 1$ , a zeroth-degree polynomial. The first row is fit by  $r = n + 1$ , a first-degree polynomial. A natural generalization of this pattern is that the second row should be fit by a second-degree polynomial: a quadratic. A quadratic requires three data points, so use  $n = 0 \dots 2$ . The polynomial that fits  $r_2$  for these points is

$$r_2(n) = \frac{1}{2}n^2 + \frac{1}{2}n + 1.$$

Does this quadratic fit the other, certain data points? For  $n = 3$ , it predicts  $r = 7$ , which is right. For  $n = 4$  it predicts  $r = 11$ , which is also right. So we

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can probably trust its prediction for  $n = 5$ , which is  $r = 16$  – in agreement with the prediction from adding numbers.

Carrying this system farther, the third row should be fit by a cubic, which needs four points for its fit. The cubic, as you can check, that fits the first four points is

$$r_3(n) = \frac{1}{6}n^3 + \dots 1$$

It predicts  $r(4) = 15$  and  $r(5) = 26$ , so once again the previous conjectures for  $r(5)$  get new support. And therefore so does the theory that predicted them.

But why is it true? Ah, that problem is left as an exercise for the reader.