

# Part 1

# Dimensions

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Mathematics developed alongside and based on the physical sciences. Famous mathematicians of the 17th and 18th centuries – Newton, the Bernoulli's, and Euler – are famous also in physics because the fields of mathematics and physics were not separated. However, mathematics now is taught mostly separate from physics, chemistry, engineering, economics, or other fields that use mathematics. Ideas and examples from these fields, before they can return to benefit mathematics, must climb a higher barrier than they had to climb centuries ago.

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Dimensions are one of those useful ideas from a foreign land. This chapter introduces the method of dimensions – more formally known as dimensional analysis – using an example from a common argument in the media, then develops the method by applying it to derivatives and integrals.

# Chapter 1

## Power of multinationals



Critics of globalization (the kinder, gentler name for imperialism) often make an argument like this one:

In Nigeria, a relatively economically strong country, the GDP is \$99 billion. The net worth of Exxon is \$119 billion. 'When multinationals have a net worth higher than the GDP of the country in which they operate, what kind of power relationship are we talking about?' asks Laura Morosini. [Source: 'Impunity for Multinationals', *ATTAC*, 11 Sept 2002, <http://www.globalpolicy.org/socecon/tncs/2002/0911impunity.htm>, retrieved 11 Sept 2006]

Before reading further, try the next problem.

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**Problem: Finding the fault.** What is the most egregious fault in the comparison between Exxon and Nigeria? It's a competitive field, but one fault stands out.

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The problems interspersed in the text are designed to make an inert object – processed dead trees with ink, also known as a textbook – into

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a conversation partner. A knowledgeable tutor is still the most effective way to learn, a fact reflected in the title of a famous paper: 'The 2 sigma problem: The search for methods of instruction as effective one-to-one tutoring' [1]. You learn when you question, wonder, and discuss; all of these activities are facilitated by a tutor. This textbook is designed for self-teaching, so it intersperses problems within the text to encourage questioning, wondering, and discussing. In short, try the problem before you read on!

The comparison between Exxon and Nigeria has many problems, and I hope that you found one. First, the comparison perhaps exaggerates Exxon's power by using its worldwide assets (net worth) rather than its assets only in Nigeria. On the other hand, Exxon can use its full international power when negotiating with Nigeria, so perhaps the worldwide assets are a fair basis for comparison.

A more serious problem is the comparison with GDP, or gross domestic product. To see the problem, look at the ingredients in how GDP is usually measured, as dollars per year. The \$99 billion for Nigeria's GDP is shorthand for \$99 billion per year. A year is an astronomical time, and its use in an economic measurement is arbitrary. Economic flows, which are a social phenomenon, should not care about how long the earth requires to travel around the sun. Suppose instead that the decade was the chosen unit of time in measuring the GDP. Then Nigeria's GDP would be (assuming it does not change from year to year) roughly \$1 trillion per decade and would be reported as \$1 trillion. Now Nigeria towers over the puny Exxon, whose assets are a mere one-tenth of this figure. To produce the opposite conclusion, measure GDP in units of dollars per week: Nigeria's GDP becomes \$2 billion per week. Now puny Nigeria stands helpless before the might of Exxon, 50-fold larger than Nigeria. Any conclusion can be produced merely by changing the units, a signal that the comparison is dubious.

The flaw in the comparison is the theme of this chapter. Assets are an amount of money – money is its dimensions – and are typically

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measured in dollars, which are the units. GDP is defined as the total goods and services sold in one year. It is a rate and has dimensions of money per time; its typical units are dollars per year. Comparing assets to GDP means comparing money to money per time. *Because the dimensions of these two quantities are not the same, the comparison is nonsense!* A similarly flawed comparison is to compare length per time (speed) with length. Listen how ridiculous it sounds: 'I walk 1.5 meters per second, much smaller than the Empire State building in New York, which is 300 meters high.' To produce the opposite conclusion, measure time in hours: 'I walk 5000 meters per hour, much larger than the Empire State building at only 300 meters.' Nonsense all around!

This example illustrates several ideas:

- *Dimensions versus units.* Dimensions are general and generic, such as money per time or length per time. Units are the instantiation of dimensions in a system of measurement. The most complete system of measurement is the System International (SI), where the unit of mass is the kilogram, the unit of time the second, and the unit of length the meter. Other examples of units are dollars per year or kilometers per year.
- *Necessary condition for a valid comparison.* In a valid comparison, the dimensions of the compared objects be identical. Do not compare apples to oranges (except in questions of taste, like 'I prefer apples to oranges.')
- *Rubbish abounds.* There's lots of rubbish out there, so keep your eyes open for it!
- *Bad argument, fine conclusion.* I agree with the conclusion of the article, that large oil companies exert massive power over poorer countries. However, as a physicist I am embarrassed by the reasoning. So I relearn a valuable lesson about theorems and proofs: judge the proof not just the theorem. You might disagree with the

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conclusion, but remember the general lesson, that a correct conclusion does not justify a dubious argument.

The following examples develop these ideas and apply them to mathematical problems of successively greater difficulty.

# Chapter 2

## Free fall



This example illustrates the missed opportunities in differential calculus where dimensions would be useful. I chose this example because an exact solution is available without needing advanced mathematical techniques. You can compare the exact, or honest solution with the approximate solution, and learn the merits of each method.

The example is a typical calculus problem:

A ball falls from a height of  $h$  **feet**. Find its speed when it hits the ground, given a gravitational acceleration of  $g$  **feet per second squared** and neglecting air resistance.

Height should be a quantity with dimensions of length, measured in units such as feet or meters. In this problem, however, its units (feet) have been separated from the  $h$ , making  $h$  a pure number, also known as a dimensionless constant. Similarly, the strength of gravity should have dimensions of length per time squared. However, the problem separates  $g$  from its units, which are feet per second squared. The formulation of the problem also makes time and speed dimensionless, although the usual practice is to solve problem that way and then explicitly restore their units. Such hacking around indicates that the method is unsound.

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Indeed, the artificial purity whereby quantities are purged of their units makes it difficult to guess the solution by using dimensions. To find the speed, you are instead forced solve this equation involving derivatives (i.e., a differential equation):

$$\frac{d^2y}{dt^2} = -g, \text{ with } y(0) = h \text{ and } \dot{y}(0) = 0,$$

where  $y(t)$  is the ball's height at  $t$  seconds,  $\dot{y}(t)$  is its velocity, and  $g$  is the strength of gravity (an acceleration). A notation note: Derivatives are usually indicated with a prime, as in  $y'(x)$ , but when the independent variable is time, a dot indicates differentiation.

Solving differential equations is a natural approach in a calculus textbook, which can hardly be faulted for keeping to techniques from calculus. However, problems from the world do not arrive limited to one field, and it is worth learning many methods to approach any problem. Therefore, this textbook divides the world into methods rather than fields: It covers the methods that I have found most useful, using examples from diverse fields. We can analyze a calculus problem, for example, with methods from physics, engineering, or pure mathematics. A complex problem leaves one unsure where or how to begin, but each method will shine a bit of light on it, and the combined light may be enough to start an analysis.

To keep this problem simple, the initial velocity is 0. If the initial velocity is not zero, the exact and dimensional-analysis solution become trickier; in the problems at the end of the chapter, you can try your hand with a non-zero initial velocity.

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**Problem: Honest solution.** Show, using calculus, that this second-order differential equation has the following solution:

$$\begin{aligned}\dot{y}(t) &= -gt, \\ y(t) &= -\frac{1}{2}gt^2 + h.\end{aligned}$$

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**Problem: Impact speed.** Show that the impact speed is  $\sqrt{2h/g}$ .

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The solution  $y(t)$  leads to the exact impact speed using the following reasoning. The ball hits the ground when  $y(t) = 0$ , which happens when  $t_0 = \sqrt{2h/g}$ . The speed after that time is

$$\dot{y}(t) = -gt_0 = -\sqrt{2gh}.$$

This derivation has many spots to make algebra mistakes: for example, forgetting to take the square root when solving for  $t_0$ , or dividing rather than multiplying by  $g$  when finding the speed. Probably I would not make those mistakes on a simple problem, but I want to develop methods for when the problems become complex and the pitfalls numerous.

Here's the same problem rewritten so that dimensions help you analyze it:

A ball falls from a height  $h$ . Find its speed when it hits the ground, given a gravitational acceleration of  $g$  and neglecting air resistance.

In this version, the dimensions of  $h$  and  $g$  are part of the quantities. The reunion helps you guess the final speed without solving differential equations. There is one caveat: It helps if you know the dimensions of the quantities. In some fields, such as electromagnetism, the systems of units are miserable. For example, what are the dimensions of magnetic field? Electric field? But learning such dimensions helps solve many electromagnetic problems. Fortunately, most problems including the free-fall problem involve quantities with simple dimensions.

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**Problem: Energy and power.** In terms of length  $L$ , mass  $M$ , and time  $T$ , find the dimensions of energy and of power.

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In the free-fall problem, the dimensions of height  $h$  are length or  $L$  for short. The dimensions of  $g$  are length per time squared or  $LT^{-2}$ , where  $T$  stands for the dimension of time, and the dimensions of speed are  $LT^{-1}$ . The speed is a function of  $g$  and  $h$ , so look for a combination of  $g$  and  $h$  with the correct dimensions.

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**Problem: A candidate.** Show that  $\sqrt{gh}$  is one combination of  $g$  and  $h$  with the dimensions of speed.

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Here are the dimensions of  $\sqrt{gh}$ :

$$\sqrt{LT^{-2} \times L} = \sqrt{L^2T^{-2}} = LT^{-1},$$

which are, as hoped, the dimensions of speed. Is  $\sqrt{gh}$  the only option?

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**Problem: Uniqueness.** Show that  $\sqrt{gh}$  is the only combination of  $g$  and  $h$  with the dimensions of speed, except for multiplication by a dimensionless constant.

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One method for showing uniqueness is to propagate constraints; the general method of constraint propagation is discussed in many references, including an old favorite [2]. If the constraints lead to only one solution, then the solution is unique. Here, the constraint is that

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the chosen combination of  $g$  and  $h$  have dimensions of speed. That constraint is composed of two smaller constraints: that the powers of L and of T match on both sides of

$$\text{speed} = \text{combination of } g \text{ and } h.$$

On the left side, speed contains  $T^{-1}$ . On the right side, only  $g$  contains T, as  $T^{-2}$ . So the combination of  $g$  and  $h$  must look like

$$\text{speed} = \sqrt{g} \times \text{function of } h.$$

The second constraint is that the powers of L match. On the left side, speed contains  $L^1$ . On the right side,  $\sqrt{g}$  contains  $L^{1/2}$ . The missing  $L^{1/2}$  must come from the function of  $h$ , so that function is  $\sqrt{h}$  times a dimensionless constant. Now the speed is determined:

$$\text{speed} = \sqrt{gh} \times \text{dimensionless constant.}$$

In a compact notation:

$$\text{speed} \sim \sqrt{gh}.$$

The  $\sim$  is a cousin of an equals sign. Useful cousins of  $=$  include, in order of increasing exactness:

- $\propto$  : equal, except perhaps for a factor perhaps with dimensions
- $\sim$  : equal, except perhaps for a factor with no dimensions
- $\approx$  : equal, except perhaps for a factor near 1

The exact result, computed earlier, is

$$\text{speed} = \sqrt{2} \times \sqrt{gh},$$

so the dimensionless factor is  $\sqrt{2}$ . The dimensions method gives almost the same answer as does solving the differential equation – and quickly with few places to make algebra mistakes. The price is not finding the factor of  $\sqrt{2}$ . This factor is close to 1, so the result from dimensions is reasonably accurate. It accounts for most variation in

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impact speed, which are due to  $h$  being able to change over a large range. To remember the exact result, remember the factor of  $\sqrt{2}$ . The rest of the result,  $\sqrt{gh}$ , is required to be that way because of dimensions. Therefore, I prefer to write the result as

$$\frac{\text{speed}}{\sqrt{gh}} = \sqrt{2}.$$

Both sides are dimensionless, and the right side isolates the one piece of information to remember.

Solving a problem by dimensions is quick and it reduces mistakes, but the method requires that quantities keep their dimensions. Hence the following rule of thumb:

Do not separate a quantity from its dimensions.

Then dimensions can guide you to correct answers and can help you check proposed answers.

# Chapter 3

## Integration

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The last example shows the value of retaining dimensions when they are specified in a problem. The same idea can be used when the dimensions are unspecified. You just have to choose the dimensions yourself. To illustrate this idea, try a Gaussian integral – so called because it is related to the normal or Gaussian distribution. Start with the simplest such integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

**Chapter 7** integrates it honestly. For now accept that the integral is  $\sqrt{\pi}$ . Dimensions do not help much with this integral because – as I show you in a few pages – exponents are always dimensionless. Therefore,  $-x^2$  is dimensionless and so is  $x$ . But dimensions help you find the value of the general Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx,$$

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where  $\alpha$  is a constant. An alternative solution is to substitute  $z$  for  $\alpha x^2$ . It gives a way to check the result derived by dimensions:

Find more than one way to solve any problem!

Using dimensions requires thinking about the quantities in the integral and the result of integrating. The extra thinking can be a disadvantage, if time is short, or an advantage because it introduces you to a problem and helps you wrap your mind around it. By way of introduction to the problem, let's figure out what form the result has. The integration variable is  $x$ ; after integrating over the integration range, the  $x$  disappears and only  $\alpha$  remains. The integral is therefore a function of  $\alpha$ :

$$f(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx.$$

The result contains only  $\alpha$  and maybe dimensionless numbers, so  $\alpha$  is the only quantity in the result that could have dimensions. For dimensional analysis to have a prayer of helping,  $\alpha$  needs dimensions. If  $\alpha$  is dimensionless, you cannot say whether, for example, the result should contain  $\alpha$  or contain  $\alpha^2$  because both choices have identical dimensions. Guessing the answer happens in three steps:

1. specifying the dimensions of  $\alpha$ ,
2. finding the dimensions of the integral, and
3. making an  $f(\alpha)$  with the dimensions of the integral.

## 3.1 Dimensions of $\alpha$

The dimensions of  $\alpha$  and  $x$  are related because the  $\alpha$  and  $x$  appear in the exponent  $e^{-\alpha x^2}$ . So let's investigate that relation dimensionally before deciding the dimensions of  $\alpha$ . An exponent, such as  $n$  in  $2^n$ , says how many times to multiply a quantity by itself:

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$$2^n = \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ terms}}$$

The notion of ‘how many times’ has no dimensions: It is a pure number. The number might be negative or be a fraction or both, but it is a pure number. Therefore:

An exponent is dimensionless.

In this problem, that rule makes  $\alpha x^2$  dimensionless. Choosing the dimensions of  $x$  determines the dimensions of  $\alpha$ ; alternatively, choosing the dimensions of  $\alpha$  determines the dimensions of  $x$ . One choice is to make  $x$  a length. This choice is natural if you think of  $x$  as representing a point on the  $x$  axis, and imagine that the  $x$  axis lies on the floor. Then  $x$  is the (signed) distance from the origin, and it has dimensions of length. The dimensions of  $\alpha$  are then  $L^{-2}$ . A convenient shorthand (also known as notation) is [quantity] to mean the dimensions of the quantity. Using that notation, we are using these dimensions:

$$\begin{aligned} [x] &= L, \\ [\alpha] &= L^{-2}. \end{aligned}$$

You need not to assign a particular dimension to  $x$  or  $\alpha$ . Instead you can leave the dimensions of  $x$  unknown, and manipulate them using  $[x]$ . For example, the requirement that  $\alpha x^2$  be dimensionless means

$$[\alpha] \times [x]^2 = 1,$$

so

$$[\alpha] = [x]^{-2}.$$

This general approach, with so many brackets, increases the notation on the page and perhaps also the opportunity to make mistakes. So I prefer to assign an intuitive dimension to  $x$ , such as a length, and to

propagate the particular choice through the problem. That propagation happens in the next step.

## 3.2 Dimensions of the integral

The second step is to find the dimensions of the integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx.$$

The dimensions of an integral depend on the dimensions of the integrand  $e^{-ax^2}$ , on the dimensions of the factor of  $dx$ , and on how integration affects dimensions. Look first at the effect of integration. An integral is the limit of a sum:

$$\lim_{\Delta x \rightarrow 0} \sum_{-\infty}^{\infty} e^{-ax^2} \Delta x \rightarrow \int_{-\infty}^{\infty} e^{-ax^2} dx.$$

The integral sign (the integration symbol) originated as an elongated 'S', standing for *Summe*, the German word for sum: Stretch an 'S', or a summation sign  $\sum$ , and it turns into an integral sign.

To deduce the consequences for our problem from this bit of history requires the fundamental principle of dimensions:

You cannot add apples to oranges.

In other words, you cannot add meters to seconds. More generally, every term in a sum has identical dimensions, and the sum has the same dimensions as any of its terms. Given the kinship between summation and integration, the dimensions of the integral are, similarly, the dimensions of one 'term' in the integral. Each term is  $e^{-ax^2} dx$  for a particular  $x$ , and the integral is the sum of these uncountably infinite number of terms. This discussion is summarized in the following principle:

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Integration does not affect dimensions: The integral of  $f(x) dx$  has the same dimensions as  $f(x) dx$ . In symbols:

$$\left[ \int f(x) dx \right] = [f(x) dx] = [f(x)] \times [dx].$$

A more cryptic but shorter statement is:

The integration symbol does not affect dimensions.

**Problem: Velocity.** Position is the integral of velocity, but position and velocity have different dimensions. How can this difference be consistent with the principle that integration does not affect dimensions?

The problem seems confusing only because I stated it so glibly. Re-stating it in slow motion resolves the paradox. Position is the (time) integral of velocity:

$$x = \int v dt,$$

where  $x$  is position and  $v$  is velocity. The dimensions of  $v$  are  $LT^{-1}$  and of  $dt$  are  $T$ , so the dimensions of  $v dt$  are  $L$ . The integral sign does change those dimensions, so the right side is a length. Good, because the left side is also a length. The resolution leads to this advice:

When finding the dimensions of an integral, do not forget the dimensions of the  $dx$ .

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In this problem, the dimensions of the general Gaussian integral are the dimensions of  $e^{-ax^2}$  times the dimensions of  $dx$ . The exponential, despite its fierce-looking exponent  $-ax^2$ , is just the dimensionless number  $e$  multiplied by itself several times. Since  $e$  has no dimensions,  $e^{\text{anything}}$  has no dimensions. So the exponential factor  $e^{-ax^2}$  contributes no dimensions to the integral.

However, the  $dx$  does contribute dimensions. To find the dimensions of  $dx$ , read  $d$  as 'a little bit of'. Then  $dx$  becomes 'a little bit of  $x$ '. Since little bit of length is a length,  $dx$  is also length. The general principle is that

$dx$  has the same dimensions as  $x$ .

Another way to arrive at this conclusion is from the definition of  $dx$ . It comes from a finite  $\Delta x$  by taking the limit  $\Delta x \rightarrow 0$  where  $\Delta x$  is, for example,  $x_{n+1} - x_n$ . The difference of two  $x$ 's has the same dimensions as  $x$ , so  $\Delta x$  and therefore  $dx$  have the same dimensions as  $x$ . A more cryptic but shorter statement (and therefore easier to work with mentally) is:

The  $d$  operation does not affect dimensions.

With all the pieces assembled, the dimensions of the integral are:

$$\left[ \int e^{-ax^2} dx \right] = \underbrace{[e^{-ax^2}]}_1 \times \underbrace{[dx]}_L = L.$$

### 3.3 Making an $f(\alpha)$ with correct dimensions

The third step is to make an  $f(\alpha)$  with the dimensions of the integral, which is a length. The dimensions of  $\alpha$  are  $L^{-2}$  to make the exponent  $\alpha x^2$  dimensionless. The only way to turn  $\alpha$  into a length is  $\alpha^{-1/2}$ , perhaps multiplied by a dimensionless constant. So

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$$f(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \sim \alpha^{-1/2}.$$

To determine the dimensionless constant hidden in the  $\sim$  symbol, set  $\alpha = 1$ :

$$f(1) = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

The two results  $f(\alpha) \sim \alpha^{-1/2}$  and  $f(1) = \sqrt{\pi}$  say that the missing dimensionless constant is  $\sqrt{\pi}$ :

$$f(\alpha) = \sqrt{\frac{\pi}{\alpha}},$$

and

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

This method of finding the integral suggests a rule of thumb:

Give dimensions to quantities with unspecified dimensions, and use the dimensions to narrow the possible results.

Setting  $\alpha = 1$  is an example of extreme- or special-cases reasoning, the topic of ??, and is slightly inconsistent with the message of this chapter. **Section 3.1** exhorted you to specify the dimensions of  $\alpha$ , which became  $L^{-2}$ . Setting  $\alpha = 1$  contradicts that specification. However, this temporary violation of good dimensional habits is useful because it shows that the missing dimensionless constant is  $\sqrt{\pi}$ . This violation is actually following another rule of thumb that is elaborated in ??:

Use special cases to augment the information from dimensions.

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This example illustrated the use of dimensions in analyzing integrals cheaply. The reverse of integration is differentiation, so the next section develops methods for cheap differentiation, including analyzing differential equations cheaply.

# Chapter 4

## Cheap differentiation

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To analyze derivatives using dimensions, start with the idea that a derivative is a quotient of  $df$  and  $dx$ :

$$f'(x) = \frac{df}{dx}.$$

You never go astray if you read  $d$  as 'a little bit of', an idea that we used to determine the dimensions of an integral. So  $df$  means 'a little bit of  $f$ ',  $dx$  means 'a little bit of  $x$ ', and

$$f'(x) = \frac{df}{dx} = \frac{\text{a little bit of } f}{\text{a little bit of } x}.$$

Using the [quantity] notation to stand for the dimensions of the quantity, the dimensions of  $f'(x)$  are:

$$[f'(x)] = \frac{[\text{a little bit of } f]}{[\text{a little bit of } x]}.$$

Since a little bit of a quantity has the same dimensions as the quantity itself,

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$$[f'(x)] = \frac{[\text{a little bit of } f]}{[\text{a little bit of } x]} = \frac{[f]}{[x]}.$$

Differentiating with respect to  $x$  is, for the purposes of dimensional analysis, equivalent to dividing by  $x$ .

In other words,  $f'(x)$  has the same dimensions as  $f/x$ .

This strange conclusion is worth testing with a familiar example. Take distance  $x$  as the function to differentiate, and time as the independent variable. The derivative of  $x(t)$  is  $\dot{x}(t) = dx/dt$ . Are the dimensions of  $\dot{x}(t)$  the same as the dimensions of  $x/t$ ? The derivative  $\dot{x}(t)$  is velocity, which has dimensions of length per time or  $LT^{-1}$ . The quotient  $x/t$  also has dimensions of length per time. So this example supports the conclusion that  $f'(x)$  and  $f/x$  have the same dimensions.

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**Problem: Double derivative.** What are the dimensions of  $f''(x)$ ?

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Since the dimensions of the derivative  $df/dx$  are the same as the dimensions of  $f/x$ , perhaps the magnitude of the derivative is similar to  $f/x$ ? Indeed it is, provided that you choose the  $f$  and  $x$  carefully. Let's use this idea to guess features of solutions to differential equations.

## 4.1 Estimating the magnitude of a derivative

Before tangling with differential equations, which are composed of derivatives, let's investigate that building block a little farther. Take a function whose derivative is easy to find:  $A(x) = x^2$ . Think of  $A$  as the area of a square with side  $x$ . Here is how to estimate the magnitude of  $A'(x)$ , by analogy with finding the dimensions of  $A'(x)$ . The dimensions of  $A'(x)$  are

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$$[A'] = \frac{[A]}{[x]}.$$

Now erase the brackets:

$$A'(x) \sim \frac{A}{x}.$$

This rewrite, which uses a twiddle ( $\sim$ ) to reflect the possible illegitimacy of removing the brackets, suggests how to estimate the magnitude of  $A'(x)$ :

$$A'(x) \sim \frac{\text{typical value of } A \text{ around } x}{\text{change in } x \text{ to change } A \text{ significantly}}.$$

Here is an estimate of  $A'(3)$  using this formula. A typical value of  $A(3)$  is 9. A significant change is not precisely defined. It might be a factor of 2 or, in many problems, a factor of  $e$ . The lack of an exact definition often does not matter, because the quantities themselves are known only approximately: a typical situation in fluid mechanics.

For definiteness, let's choose a factor of 2 change as the standard for a significant change. For  $A$  to grow by a factor of 2,  $x$  needs to grow by a factor of  $\sqrt{2}$  or roughly 1.4. So when  $x$  is roughly  $3 \times 1.4 = 4.2$ , the area  $A$  has changed significantly. The  $x$  interval is  $[3, 4.2]$ , which means  $x$  needs to change by 1.2. The estimate for  $A'(3)$  is:

$$A'(3) \sim \frac{9}{1.2} = 7.5.$$

This estimate is inaccurate by only 25%: The exact derivative is 6. Another origin of this method is the definition of derivative:

$$A'(x) = \frac{dA}{dx} \approx \frac{\Delta A}{\Delta x},$$

where the approximation becomes exact as  $\Delta x \rightarrow 0$ . Instead of taking  $\Delta x \rightarrow 0$ , which makes  $\Delta A \rightarrow 0$ , the significant-change approximation chooses  $\Delta x$  to be large enough to change  $A$  significantly. This method is the opposite of calculus. However, global values, such as the change in  $x$  required to change  $A$  significantly, are often easier to guess than are point values like  $\Delta A$  for a given tiny  $\Delta x$ . This inexact

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method might help you make progress when exact method are too difficult to use.

**Problem: Different location.** Estimate  $A'(5)$  using this method and compare the estimate with the true derivative.

Why bother with this method when the exact derivative,  $A'(x) = 2x$ , is easy to compute? Because it illustrates the method in an example where we know the answer, so we develop experience before using the method in complex problems. An example of a complex problem is the Navier–Stokes equations of fluid mechanics:

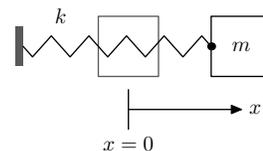
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}.$$

This equation starts in ???. Rather than rush into an analysis now, the next example builds a halfway house using a spring–mass system: It is not one-half as complex as Navier–Stokes but it develops the main ideas.

## 4.2 Spring–mass system

Here is the differential equation describing the oscillations of a mass connected to a spring:

$$m \frac{d^2 x}{dt^2} + kx = 0,$$



where  $m$  is the mass,  $x$  is its position,  $t$  is time, and  $k$  is the stiffness of the spring, also known as the spring constant. An ideal spring exerts a force proportional to its extension  $x$ , and the constant of proportionality is the stiffness:

$$k \equiv \frac{\text{force}}{\text{extension}}.$$

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**Problem: Dimensions of spring constant.** What are the dimensions of  $k$ ?

In the first term, the second derivative  $d^2x/dt^2$  is the acceleration  $a$  of the mass, so  $m(d^2x/dt^2)$  is  $ma$ , which is a force: Remember Newton's second law  $F = ma$ . The second term,  $kx$ , is the force exerted by the spring. Working out the meaning of the terms also checks that their dimensions are the same (here, dimensions of force). The equation is therefore dimensionally correct, an important check on any equation that you see.

**Problem: Signs.** In the spring equation, should the  $kx$  term be  $+kx$ , as it is written now, or should it be  $-kx$ ?

The equation contains a second derivative, whose dimensions we need for the rest of the analysis.

**Problem: Second derivative.** Which, if any, of these choices is correct:

$$\left[ \frac{d^2x}{dt^2} \right] = \begin{cases} [x^2] / [t^2] \\ \text{or} \\ [x] / [t^2]? \end{cases}$$

Both choices are plausible. The numerator contains  $d^2$  and  $x$ , and the exponent of 2 argues for the first choice, because it too has an exponent of 2 near an  $x$ . To decide between the choices, use the principle that  $d$  means 'a little bit of'. Then  $dx$  means 'a little bit of  $x$ ', and

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$d^2x = d(dx)$  means ‘a little bit of a little bit of  $x$ ’. The numerator has the same dimensions as  $x$ , and has nothing to do with  $x^2$ . The other half of the second derivative is the denominator  $dt^2$ . It is a lazy way of writing  $(dt)^2$ , and each  $dt$  has the same dimensions as  $t$ . So the denominator has dimensions  $[t^2]$ . The dimensions of  $d^2x/dt^2$  are  $[x]/[t^2]$ .

Another way of expressing the same idea is that differentiation is a linear operation: that the derivative of  $x + y$  is  $\dot{x} + \dot{y}$ . If the second derivative of  $x$  were related to  $x^2$ , then the second derivative of  $3x$  would involve  $9\ddot{x}$ . However, because differentiation is linear, the first derivative of  $3x$  is  $3\dot{x}$ , its second derivative is  $3\ddot{x}$  rather than  $9\ddot{x}$ .

Just as the dimensions of  $d^2x/dt^2$  are  $[x]/[t^2]$ , the magnitude of  $d^2x/dt^2$  is roughly  $x/t^2$ , with a skillful choice of  $x$  and  $t$ .

**Problem: Simple example.** Try the approximation that  $d^2x/dt^2 \sim x/t^2$  on the function  $x = t^2$ . How does the approximation compare to the exact second derivative?

When  $x = t^2$ , the true second derivative is 2 whereas the approximation, applied without thought about what  $x$  or  $t$  to use, gives  $d^2x/dt^2 \sim 1$ , a reasonable estimate. In more complex examples, more thought is needed. In this problem, all the positions  $x(t)$  are scaled relative to the initial displacement  $x_0$  (also known as the amplitude). If you double  $x_0$ , you double all the positions. So a reasonable typical value of  $x$  is the initial displacement  $x_0$ . For  $t$ , which is in the denominator, use a time  $\tau$  in which the numerator  $x$  changes significantly. That time, the **characteristic time**, is related to the oscillation period, a relation that will become clearer after we find  $\tau$  and compare the approximate and exact solutions.

The plan is first to approximate the derivative  $d^2x/dt^2$  in the differential equation, turning the differential equation into an algebraic equation; and then to solve the algebraic equation to find  $\tau$ . To approximate the second derivative, start with the first derivative:

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$$\frac{dx}{dt} \sim \frac{\text{typical } x}{\tau} \sim \frac{x_0}{\tau}.$$

Then

$$\frac{d^2x}{dt^2} = \frac{d}{dt} \left( \frac{dx}{dt} \right) \sim \frac{1}{\tau} \frac{x_0}{\tau} = \frac{x_0}{\tau^2},$$

which is an algebraic expression that is easier to handle than a derivative.

Now we can estimate the two terms in the differential equation, which are  $m(d^2x/dt^2)$  and  $kx$ . A typical value of  $kx$  is  $k$  times a typical value of  $x$ , so the estimates are

$$kx \sim kx_0,$$

$$m \frac{d^2x}{dt^2} \sim m \frac{x_0}{\tau^2}.$$

The differential equation

$$m \frac{d^2x}{dt^2} + kx = 0$$

says that the two terms add to zero. Therefore their magnitudes are comparable. Using the estimates produces this relation for the characteristic time  $\tau$ :

$$m \frac{x_0}{\tau^2} \sim kx_0.$$

Both sides contain one power of the amplitude  $x_0$ , so  $x_0$  divides out: That cancellation always happens in a *linear* differential equation. With  $x_0$  gone, it cannot affect the time constant  $\tau$ . This independence is worth stating in words:

In ideal spring motion – so-called simple harmonic motion – the oscillation period is *independent* of amplitude.

After cancelling the  $x_0$ , the leftover is

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$$\frac{m}{\tau^2} \sim k,$$

which gives the characteristic time  $\tau \sim \sqrt{m/k}$ . The position changes significantly on this timescale. The reciprocal of a time is a frequency, so the position changes significantly at a frequency  $\sqrt{k/m}$ .

To see what kind of frequency it is, cheat by looking at the exact solution. These examples are chosen to enable this kind of cheating. The exact solution is

$$x = x_0 \cos \omega t,$$

where  $\omega = \sqrt{k/m}$  is the so-called angular frequency.

**Problem: Verification.** Substitute  $x = x_0 \cos \omega t$  into the spring differential equation to show that  $\omega = \sqrt{k/m}$ .

**Problem: Dimensions in a cosine.** What are the dimensions of  $\omega t$ ?

The dimensions of  $\sqrt{k/m}$  are  $T^{-1}$ , so  $\omega t$ , the argument of the cosine, is dimensionless. Like the exponential function  $e^x$ , the cosine and sine functions require a dimensionless argument.

The exact solution for  $x$  gives an alternative way to find the characteristic time. When  $t = 0$  then  $\omega t = 0$  and the mass is at  $x = x_0$ . When  $\omega t = \pi/2$ , the mass reaches the equilibrium position  $x = 0$ , at which point  $x$  has changed more than significantly: A significant change would be from  $x_0$  to, say,  $x/2$  whereas here  $x$  went all the way to 0. The quotient  $\pi/2$  is slightly larger than 1, so a reasonable measure of when  $x$  has changed significantly is when  $\omega t = 1$ : in other words, when the oscillation has advanced by 1 radian. The time for this change is  $t = \sqrt{k/m}$ . So this is the characteristic time based on the

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exact solution. And voilà, it is also the characteristic time that resulted from turning the differential equation into an algebraic equation. This comparison gives a useful rule of thumb:

In an oscillating system, the system changes significantly when the cosines or sines advance by 1 radian.

Besides illustrating this rule of thumb, the example shows how to turn differential equations into algebraic equations. This conversion has a cost – accuracy – and a great benefit: that algebra is easier than calculus. This problem-solving habit is worth a box:

Turn every differential equation into an algebraic equation.

# Chapter 5

## What you have learned



This chapter introduced several problem-solving techniques:

- Preserve dimensions in quantities with dimensions: Do not write ‘g meters per second squared’; write  $g$ .
- Choose dimensions for quantities with arbitrary dimensions, like for  $x$  and  $a$  in

$$\int_{-\infty}^{\infty} e^{-ax^2} dx.$$

- Turn every differential equation into an algebraic equation.

This chapter also introduced several problem-solving rules of thumb and principles:

- Exponents are dimensionless – as are the arguments of most elementary functions (such as sine and cosine).
- You cannot add apples to oranges: Every term in an equation or sum has identical dimensions.
- The dimensions of an integral are the dimensions of everything inside it, including the  $dx$ . This principle helps you guess integrals such as the general Gaussian integral with  $-ax^2$  in the exponent.

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- The dimensions of a derivative  $f'(x)$  are the dimensions of  $f/x$ . This principle helps reconstruct formulas based on derivatives, such as Taylor or MacLaurin series.
- The magnitude of  $df/dx$  is roughly

$$\frac{\text{typical size of } f}{x \text{ interval over which } f \text{ changes significantly}}$$

# Chapter 6

## Problems



**Arcsine integral.** Given this integral:

$$\int \sqrt{1-x^2} dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2},$$

use dimensions to find  $\int \sqrt{1-3x^2} dx$ .

**Free fall with initial velocity.** The ball in **Chapter 2** was released from rest. What if it had an initial velocity  $v_0$  (where positive  $v_0$  means an upward throw)? Solve the differential equation to find its impact velocity  $v_{\text{final}}$ . You can approximate this result but doing so requires learning another method, extreme cases, which is the subject of ??.

**Integral of an exponential.** Find

$$\int_0^{\infty} e^{-ax} dx$$

using dimensions.

**Arctangent.** Using dimensions, find

$$\int \frac{dx}{x^2 + a^2}.$$

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A useful result is the following:

$$\int \frac{dx}{x^2 + 1} = \tan^{-1} x + C.$$