MIT 18.06 Makeup Exam 3 Solutions, Spring 2022
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Problem 1 (8+7+8+15 points):

A is a **Hermitian** matrix with eigenvectors (each normalized to length $\|x_k\| = 1$) given by the columns of the following matrix (shown to 3 decimal places):

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \approx \begin{pmatrix} 0.236 & 0.247 & 0.676 & 0.154 & 0.634 \\ -0.548 & -0.495 & 0.094 & 0.653 & 0.138 \\ 0.765 & -0.582 & -0.164 & 0.211 & 0.066 \\ 0.117 & -0.078 & 0.655 & 0.100 & -0.736 \\ -0.211 & -0.591 & 0.279 & -0.703 & 0.182 \end{pmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_3 = 3$, $\lambda_4 = 2$, and $\lambda_5 = 1$. Using this matrix A, we solve a system of ODEs:

$$\frac{dy}{dt} - \alpha y = Ay$$

for some initial condition $y(0)$ to find $y(t)$ and some **real or complex** number α.

(a) What are the eigenvalues of $X^T X$?

(b) Write the solution as $y(t) = e^{Bt} y(0)$ for some matrix B: give a formula for B in terms of A and α.

(c) Give a value of α that would cause the solution $y(t)$ to **decay to zero** for *all* initial conditions $x(t)$.

(d) For $\alpha = -5$, give a **good approximation** for $y(100)$ if

$$y(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

You can leave your solution in the form of some vector times some coefficient(s) **without carrying out the explicit multiplications**, but give all the numbers in your vector and coefficients to 3 decimal digits.

Solution

(a) Since A is Hermitian and the eigenvalues are distinct, the corresponding eigenvectors are orthogonal, and furthermore you were told that they are normalized to unit length, and so the columns of X are **orthonormal**. Hence $X^T X = I$, which has only one eigenvalue $\boxed{\lambda = 1}$ (with multiplicity 5).

(b) $\frac{dy}{dt} = (A + \alpha I)y$ so $\boxed{B = A + \alpha I}$.

(c) The solutions $e^{Bt}y(0)$ are always decaying if the eigenvalues of B have **negative real parts**. Since the eigenvalues of $B = A + \alpha I$ are $\lambda_k + \alpha$ where the λ_k are the given eigenvalues of A then **any** α **with** $\Re[\alpha] < -5$ **would suffice.** For example, $\alpha = -6$ or $\alpha = -6 + i$.

(d) For $\alpha = -5$, the eigenvalues of B are $0, -1, -2, -3, -4$, so for a large t the eigenvalues are dominated by the x_1 component, whereas the other eigenvector components decay exponentially to zero. More explicitly, imagine expanding $y(0)$ in the basis of eigenvectors:

$$y(0) = Xc = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5,$$

in which case the solution just multiplies each term by the corresponding $e^{\lambda t}$:

$$y(t) = c_1x_1 + c_2e^{-t}x_2 + c_3e^{-2t}x_3 + c_4e^{-3t}x_4 + c_5e^{-4t}x_5.$$

For $t = 100$, the decaying terms are negligible and we get

$$y(100) \approx c_1x_1.$$

But, since this is an orthonormal basis, we can get c_1 by projection:

$$c_1 = x_1^T y(0) = 0.117$$

and hence

$$\boxed{y(100) \approx 0.117x_1 = 0.117 \begin{pmatrix} 0.236 \\ -0.548 \\ 0.765 \\ 0.117 \\ -0.211 \end{pmatrix}}.$$

Notice that essentially **no arithmetic** was required. If you tried to solve $Xc = y(0)$ for c by Gaussian elimination, without exploiting the fact that X is orthonormal (so $c = X^T y(0)$), you would have had a difficult time!

Problem 2 (8+8+8+8 points):

A is the matrix

$$A = \begin{pmatrix} -1 & 18 & 4 & 3 & 17 \\ & 3 & 3 & 5 & 1 \\ & & 0 & -1 & 2 \\ & & & 2 & 4 \\ & & & & 1 \end{pmatrix}.$$

(a) What are the eigenvalues of A?

(b) What is $\det((A+2I)^2)$?

(c) If you solve $\frac{dx}{dt} = -A^T A x$ for $x(t)$ given some randomly chosen initial condition $x(0)$, would you typically expect the solutions $x(t)$ to **diverge**, **decay to zero**, **approach a nonzero constant vector**, or **oscillate forever** as $t \to \infty$?

(d) If you compute $x_n = (\frac{1}{3}A - \frac{2}{3}I)^n x$ for some randomly chosen initial vector x_0, would you typically x_n to **diverge**, **decay to zero**, **approach a nonzero constant vector**, or **oscillate forever** as $n \to \infty$?

Solution

(a) The matrix A is upper triangular and so you can read the eigenvalues off of the diagonal entries: $\boxed{\lambda = -1, 3, 0, 2, 1}$.

(b) The determinant is the product of the eigenvalues, and the eigenvalues of $(A+2I)^2$ are $(\lambda+2)^2 = 1, 25, 4, 16, 9$. Their product is $1 \times 25 \times 4 \times 16 \times 9 = 100 \times (160-16) = 100 \times 144 = \boxed{14400}$. (This is a lot easier than computing $(A+2I)^2$ first!)

(c) This hinges on the **signs of the (real) eigenvalues** of $-A^T A$. Any matrix of the form $-A^T A$ is negative semidefinite for any A, so its eigenvalues can be ≤ 0. Whether it has a 0 eigenvalue depends on $N(-A^T A) = N(A^T A) = N(A)$, but we know that A has an eigenvalue $\lambda = 0$ from above and so it must have a nonzero vector in its nullspace. Hence $-A^T A$ must **also** have a zero eigenvalue. Hence, the solutions $x(t) = e^{-A^T A t} x(0)$, if we expand in the basis of eigenvectors of $-A^T A$, contain terms that decay exponentially (corresponding to the negative eigenvalues), but also one term that is constant (the $\lambda = 0$ term). Hence, we would typically expect the solutions to **approach a nonzero constant vector** as $t \to \infty$. (The only exceptions would arise when $x(0)$ is orthogonal to the $\lambda = 0$ eigenvector, in which case the solution would decay to zero.)

(d) This kind of matrix-power recurrence depends on the **magnitudes** of the eigenvalues of $\frac{1}{3}A - \frac{2}{3}I$, which are $\frac{\lambda-2}{3} = -1, \frac{1}{3}, -\frac{2}{3}, 0, -\frac{1}{3}$. All of these have magnitudes < 1 except for -1 . So, if you expand x in the basis of eigenvectors of A (which is diagonalizable since its eigenvalues are distinct), then the terms in $x_n = (\frac{1}{3}A - \frac{2}{3}I)^n x$ will go as $(-1)^n, (\frac{1}{3})^n, (-\frac{2}{3})^n, 0^n$, and $(-\frac{1}{3})^n$. For large n, this is dominated by $(-1)^n$, which **oscillates forever**.

Problem 3 (6+6+6+6+6+6 points):

For each of the following, say what **must** be true of the **eigenvalues** λ of A (which you can assume is **diagonalizable**) if:

(a) $\|e^{(A-I)t}x\| \to \infty$ for **some** x as $t \to \infty$.

(b) $\|e^{(A-I)t}x\| \to \infty$ for **all** $x \neq 0$ as $t \to \infty$.

(c) $\|(I + A^2)^n x\|$ does *not* diverge for **any** x as $n \to \infty$.

(d) A is a Markov matrix but $A^n x$ does **not** approach a constant vector as $n \to \infty$ for some initial x.

(e) A^2 is Hermitian.

Solution

(a) $A - I$ must have at least one eigenvalue with a positive real part to get a diverging solution, so A must have **at least one eigenvalue with a real part** > 1**.**

(b) To get *only* diverging solutions here, *every* eigenvalue of $A-I$ must have a positive real part (since we can just choose x to be any of the eigenvectors). So **all of the eigenvalues of** A **must have real parts** > 1**.**

(c) To get *no* diverging solutions, then $I + A^2$ must have eigenvalues with magnitude ≤ 1. If λ is an eigenvalue of A, then $I + A^2$ has an eigenvalue $1 + \lambda^2$. Hence, we must have $|1 + \lambda^2| \leq 1$ for **every** eigenvalue of A.

(d) If it does *not* approach a constant vector, then the only other possibility is an *oscillating* solution. (Markov matrices *cannot* have diverging $A^n x$ because all their eigenvalues have magnitude ≤ 1). This arises when A has **at least one** eigenvalue $\lambda \neq 1$ with $|\lambda| = 1$ (i.e. somewhere on the complex unit circle but $\neq 1$, such as -1 or i).

(e) The eigenvalues of A^2 must be purely real, but these are the squares λ^2 of the eigenvalues of A. So, each eigenvalue λ of A must be the $\pm$**square root of a real number**, which is **either purely real or purely imaginary (with either sign)**.