18.06 Professor Edelman Quiz 1 February 28, 2020

Your easy to read printed name is: ____________________

(We need your name on every page for gradescope.)

(Exam ends at 11:55am.)

Please circle your recitation:

(1)	T 10	36-155	Yau Wing Li
(2)	T 10	36-153	Sung Woo Jeong
(3)	T 11	36-153	Sung Woo Jeon
(4)	T 12	2-146	Yau Wing Li
(5)	T 12	2-136	James Tao
(6)	T 1	2-136	James Tao
(7)	T 1	2-142	Kai Huang
(8)	T 2	2-136	Kai Huang
(9)	T 3	2-136	Yu Pan

Important Instructions: We will be using Gradescope. Please write on one side only of a page. If you need extra pages, please write continued, and on the extra pages clearly label with problem number and letter.

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1 (20 pts.)

For each set below, decide if it is or is not a vector space. Explain briefly why or why not.

1. (a) (4 pts.) All 10×2 tall-skinny orthogonal matrices.

Answer: No 0 times a tall-skinny orthogonal matrix is not a tall-skinny orthogonal matrix any more.

1. (b) (4 pts.) All polynomials in x that are 0 at $x = 1806$ and $x = 2000$.

Answer: Yes For any function $f(x), g(x)$ that are 0 when $x = 1806$ and 2000, $kf(x)+lg(x)$, for any $k, l \in \mathbb{R}$ would be 0 when $x = 1806$ and 2000.

1. (c) (4 pts.) All $(n+1) \times n$ matrices of the form $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & v_n \end{pmatrix}$.

Answer: Yes Suppose A, B are matrices of the required form with the shifted diagonal being $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$. Then for any $k, l \in \mathbb{R}$, the matrix $kA + lB$ is also of the required form with the shifted diagonal being $ka_1 + lb_1, \cdots, ka_n + lb_n$.

1. (d) (4 pts.) All functions $f(x)$ of the form $c_1e^x + c_2e^{-x}$, where c_1 and c_2 are real scalars.

Answer: Yes If $f(x) = c_1e^x + c_2e^{-x}$ and $g(x) = d_1e^x + d_2e^{-x}$, then, for any $k, l \in \mathbb{R}$, $kf(x) + lg(x) = (kc_1 + ld_1)e^x + (kc_2 + ld_2)e^{-x}$.

1. (e) (4 pts.) All 5×5 symmetric matrices A (meaning $A = A^T$).

Answer: Yes If A and B are symmetric matrices, so is $kA + lB$, for any $k, l \in \mathbb{R}$, since $(kA + lB)^T = kA^T + lB^T = kA + lB$.

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2 (15 pts.)

A researcher measures the temperature at n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$. The temperatures measured are $f_1, f_2, \ldots, f_n$ respectively. This researcher wants to find a best set of coefficients a, b, c, d, e, g to fit a function of the form

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + g$$

to the data.

2. (a) (8 pts.) Set up an equation of the form $Ax \approx b$ that represents this researcher's problem.

Answer: $$\begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & x_ny_n & y_n^2 & x_n & y_n & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ g \end{pmatrix} \approx \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

2. (b) (4 pts.) Suppose the matrix A in the above can be written as $A = QR$, where Q is tall-skinny orthogonal and R is invertible and upper triangular. What are the dimensions of Q? and the dimensions of R?

Answer: Q is a $n \times 6$ matrix and R is a 6×6 matrix.

2. (c) (3 pts.) Write the solution to the best set of coefficients in terms of possibly Q,Q^T,R, or R^{-1} and the given temperatures.

Answer: $$\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \\ \hat{e} \\ \hat{g} \end{pmatrix} = R^{-1}Q^T \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

3 (15 pts.)

How many parameters are needed? We are looking for the minimum required to specify the object. Briefly explain.

3. (a) (5 pts.) The "one cold" vector has $(n-1)$ elements 1 and the remaining one 0. How many parameters are needed to represent a "one cold" vector when n is not fixed in advance?

Answer: $\boxed{2.}$ One is for the size n of the vector and one is for the location of 0.

3. (b) (5 pts.) How many parameters are required to represent a rank-1 two by two matrix? (Possible hint: it may be easier to see the correct answer with the svd, though this problem can be done without the svd if you think carefully.)

Answer: $\boxed{3.}$ A two by two rank-1 matrix A has the SVD as

$$A = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} (a) \begin{pmatrix} \cos(\phi) & \sin(\phi) \end{pmatrix}.$$

So we need three parameters θ, a and ϕ.

3. (c) (5 pts.) An anti-symmetric matrix is one where $A^T = -A$. How many parameters are required to represent a 4 x 4 anti-symmetric matrix?

Answer: $\boxed{6.}$ A 4×4 anti-symmetric matrix is of the form

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

So six parameters are needed.

4 (10 pts.)

A square matrix A has first column and last column all ones. Why can't it have an inverse?

Answer: Suppose A is an $n \times n$ matrix. Consider a matrix E which are 1 on the diagonal, is -1 as $(1, n)$-entry and is 0 otherwise. Note that E is an invertible matrix. If A is invertible, then $B = AE$ is invertible. However B has the last column as 0 and thus can not be invertible. Therefore A is not invertible.

5 (20 pts.)

The rank-r SVD of

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 5 & 12 & 2 \end{pmatrix}$$

is numerically computed with Julia to be $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} -0.203600 & -0.585801 \\ -0.543021 & -0.599144 \\ -0.814662 & 0.545769 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 6.136942826453964 & \\ & 0.7740001393771697 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.365991 & 0.446524 \\ -0.912869 & -0.00172137 \\ -0.180887 & -0.89477 \end{pmatrix}$$

5. (a) (5 pts.) What is the rank of A?

Answer: 2.

5. (b) (6 pts.) A linearly transforms the unit sphere $\{x : \|x\| = 1\}$ into a filled ellipse in a plane, not an ellipsoid. What are the lengths of the semi-axes of this ellipse?

Answer: $6.136942826453964, 0.7740001393771697.$

5. (c) (9 pts.) Circle the (chopped) numbers in U, Σ, V below that would figure in the best rank-1 approximation to A.

$$U = \begin{pmatrix} -0.2036 & -0.5858 \\ -0.5430 & -0.5991 \\ -0.8147 & 0.5458 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 6.1369 & \\ & 0.7740 \end{pmatrix} \quad V = \begin{pmatrix} -0.3659 & 0.4465 \\ -0.9129 & -0.0017 \\ -0.1809 & -0.8948 \end{pmatrix}$$

Answer: The best rank 1 approximation is $u_1 \sigma v_1^T$, where u_1, v_1 are the first column of U and V, and $\sigma = 6.1369$.

6 (20 pts.)

The matrix $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$ and the matrix $F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 6 & 0 & 1 \end{pmatrix}$.

Ideally without working too hard, calculate $E^{-1}F^{-1}$.

Answer: $E = E_4 E_3 E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$$F = E_6 E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So $E^{-1}F^{-1} = E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -6 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & -5 & 1 & 0 \\ -4 & -6 & 0 & 1 \end{pmatrix}.$$

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