Problem Set 6 Solution

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Prob 1 (GS p.161 Problem 24) Give examples of matrices A for which the number of solutions to $Ax = b$ is

(a) 0 or 1, depending on b

Example: $A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Check: $Ax = \begin{pmatrix} 0 \\ x \end{pmatrix}$, so that $Ax = b$ has a unique solution if $b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$ and no solution if $b_1 \neq 0$.

(b) ∞, regardless of b

Example: $A = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Check: $Ax = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1$, so that $Ax = b$ has infinitely many solutions $x_1 = b, x_2 \in \mathbb{R}$.

(c) 0 or ∞, depending on b

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Check: $Ax = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, so that $Ax = b$ has infinitely many solutions $x_1 = b, x_2 \in \mathbb{R}$ if $b_2 = 0$, and no solution if $b_2 \neq 0$.

(d) 1, regardless of b

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Check: A is a 2 by 2 invertible matrix, so any equation $Ax = b$ has a unique solution $x = A^{-1}b$.

Prob 2 (Inspired by GS p163 Problem 34). Suppose A is 3 by 4 and the nullspace consists of multiples of $s = (2, 3, 1, 0)$.

(a) What are the dimensions of the four fundamental subspaces?

Since $\text{null}(A) = \{\lambda s \mid \lambda \in \mathbb{R}\}$, so $\dim \text{null}(A) = 1$. Recall that for an $m \times n$ matrix A, $\dim \text{null}(A) = n - \text{rank}(A)$, so we know $\text{rank}(A) = 3$.

Recall that $\dim \text{col}(A) = \dim \text{row}(A) = \dim \text{rank}(A), \dim \text{null}(A^T) = m - \dim \text{rank}(A)$. So $\dim \text{col}(A) = \dim \text{row}(A) = 3, \dim \text{null}(A) = 1, \dim \text{null}(A^T) = 0$.

(b) How do you know that $Ax = b$ can be solved for all b?

The equation $Ax = b$ can be solved is equivalent to say $b \in \text{col}(A)$. We know $\dim \text{col}(A) = 3$ and $\text{col}(A) \subset \mathbb{R}^3$. So $\text{col}(A) = \mathbb{R}^3$. In other words, any $b \in \mathbb{R}^3$ lies inside $\text{col}(A)$, so $Ax = b$ can be solved.

Prob 3 (GS p.177 Problem 18) Suppose $v_1, v_2, ..., v_6$ are six vectors in $\mathbb{R}^4$.

(a) Those vectors (do)(do not)(**might not**) span $\mathbb{R}^4$.

(b) Those vectors (are)(**are not**) (might be) linearly independent.

Consider the 4 by 6 matrix A whose column vectors are precisely v_i. If v_i are linearly independent, then $\dim \text{col}(A) = 6$. But $\dim \text{col}(A) = \dim \text{rank}(A) = \dim \text{row}(A) \leq 4$. This is a contradiction.

(c) Any four of those vectores (are)(are not)(**might be**) a basis for $\mathbb{R}^4$.

Prob 4 (Inspired by GS p.192 Problem 21). Under what possible conditions is the matrix $A = uv^T + wz^T$ not of rank 2?

We have $\text{col}(uv^T) \subset \text{col}(u) = \{\lambda u \mid \lambda \in \mathbb{R}\}$, $\text{col}(wz^T) \subset \text{col}(w) = \{\lambda w \mid \lambda \in \mathbb{R}\}$. If $\{\lambda u \mid \lambda \in \mathbb{R}\} = \{\lambda w \mid \lambda \in \mathbb{R}\}$. Then $\text{col}(A) = \{\lambda w \mid \lambda \in \mathbb{R}\}$, so $\text{rank}(A) \leq 1$(could be 0 if $u = w = 0$). Also if v, z are colinear, same argument shows $\text{rank}(A) \leq 1$.

If both of these conditions are not true, then $\text{col}(A) = \{\lambda u + \mu w \mid \lambda, \mu \in \mathbb{R}\}$, so $\text{rank}(A) = \dim \text{col}(A) = 2$.

So the condition is u and w are colinear or v and z are colinear.

Prob 5 (GS p.203 Inspired by Problem 10). If A is symmetric, why is the column space perpendicular to the nullspace?

We know the $\text{row}(A) \perp \text{null}(A)$. Since $A^T = A$, so $\text{col}(A) = \text{row}(A^T) = \text{row}(A)$, so $\text{col}(A) \perp \text{null}(A)$.

Prob 6 (GS p.202 Problem 4). If $AB = 0$ then the columns of B are in the null space of A. The rows of A are in the left null space of B. With $AB = 0$, why can't A and B be 3x3 matrices of rank 2?

Since we have $\mathrm{col}(B) \subset \mathrm{null}(A)$, so $\dim\mathrm{col}(B) \leq \mathrm{null}(A)$. Recall that $\dim\mathrm{col}(B) = \mathrm{rank}(B), \dim\mathrm{null}(A) = 3 - \mathrm{rank}(A)$, so $\mathrm{rank}(B) \leq 3 - \mathrm{rank}(A)$. Hence $\mathrm{rank}(A) + \mathrm{rank}(B) \leq 3$. So they cannot both have rank 2.

Prob 7 (GS p.204 Problem 24). Suppose an $n \times n$ matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

Let A be $\begin{pmatrix} v_1^T \\ v_2^T \\ \dots \\ v_n^T \end{pmatrix}$, where v_i^T is its i-th row vector. Let A^{-1} be $\begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}$, where u_i is its i-th column vector. Then $AA^{-1} = I$ gives us $v_i^T u_1 = 0$ if $i \neq 1$ and 1 if $i = 1$. So the first column of A^{-1} is orthogonal to the space spanned by the 2nd,3rd,...,n-th rows of A.

Prob 8. Given an $m \times n$ matrix A. What are the possible dimensions?

(A) $\dim(\mathrm{col}(A)) = \underline{\mathrm{rank}(A) = 0, 1, \dots, \min(m, n)}$

(B) $\dim(\mathrm{row}(A)) + \dim(\mathrm{null}(A)) = \underline{n}$

(C) the sum of the dimensions of the four fundamental subspaces $= \underline{m + n}$

(D) $\dim(\mathrm{col}(A)) + \dim(\mathrm{row}(A)) = \underline{2\mathrm{rank}(A) = 0, 2, 4, \dots, 2\min(m, n)}$

Prob 9. Suppose $y_1(x), y_2(x), y_3(x), y_4(x)$ are four non-zero polynomials of degree at most 2. (This means the functions have the form $ax + bx + c$, where at least one of the coefficients is nonzero.) What possibilities are there in the dimension of the vector space spanned by $y_1(x), y_2(x), y_3(x), y_4(x)$? Give examples for each possibility and explain briefly why no other dimension can happen.

Since $ax^2 + bx + c = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$, we can view y_i as a triple $\begin{pmatrix} a_i & b_i & c_i \end{pmatrix}$. So they lies in $\mathbb{R}^3$. So the subspace spanned by them is at most 3 dimensional.

The dimension can not be 0. Because all y_i are nonzero.

The dimension can be 1. For example, $y_1 = y_2 = y_3 = y_4 = 1$.

The dimension can be 2. For example, $y_1 = y_2 = y_3 = 1, y_4 = x$.

The dimension can be 3. For example, $y_1 = y_2 = 1, y_3 = x, y_4 = x^2$.

Prob 10. A reflector is defined as a matrix of the form $Q = I - 2uu^T$ where $||u|| = 1$.

(A) Show that a reflector is orthogonal by showing that Q is symmetric and $Q^2 = I$.

$Q^T = (I - 2uu^T)^T = I - 2(uu^T)^T = I - 2uu^T = Q$. So Q is symmetric.

$Q^2 = (I - 2uu^T)(I - 2uu^T) = I - 4uu^T + 4uu^T uu^T$. Notice that $1 = ||u||^2 = u^T u$, so $4u(u^T u)u^T = 4uu^T$, so $Q^2 = I$.

(B) Explain briefly why this makes Q orthogonal.

We have $QQ^T = QQ = I$. By definition of orthogonal matrix, we know Q is orthogonal.