pset2

September 7, 2017

1 Problem Set 2

Due Wednesday, 2/22 at 11am.

1.1 Problem 1

(From Strang, section 2.2, problem 25.)

$$A = \begin{pmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{pmatrix}?$$

For which three numbers a will elimination fail to give three pivots for this matrix? That is, for which values of a is this matrix *singular*?

1.2 Problem 2

Suppose we *already know* the inverse A^{-1} of a $m \times m$ matrix A. Now, we want to find the inverse $(A+uv^T)^{-1}$, where u and v are m-component column vectors. Ideally, we'd like to do this without re-doing the whole matrix-inversion process!

1.2.1 part (a)

Find the scalar (number) α so that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{\alpha}$$

(Hint: if you see an expression like x^TBy, realize that this is just a scalar and can be commuted with any other matrix/vector operations.)

1.2.2 part (b)

Because matrix multiplication is associative, we can compute $A^{-1}uv^TA^{-1}$ from above in different orders:

$$A^{-1}(u(v^TA^{-1})) = A^{-1}((uv^T)A^{-1}) = (A^{-1}u)(v^TA^{-1})$$

If $m = 5$ (i.e. A is a 5×5 matrix and u and v are 5-component column vectors) compute *how many scalar multiplications* (multiplications of numbers) are required if we do the products indicated by the parentheses for these three different parenthesizations of $A^{-1}uv^TA^{-1}$, assuming you are given A^{-1}, u, and v (and that all matrix entries are nonzero so you can't skip any multiplies). (You *don't* need to actually do the matrix products, just work out how many multiplications they would require!)

Which order (parenthesization) would you choose to calculate $A^{-1}uv^TA^{-1}$ for your $(A + uv^T)^{-1}$ expression in part (a) in order to minimize your work?

For example, the outer product uv^T produces an $m \times m$ matrix, whose (i, j) entry is u_iv_j. So, there is one multiplication per entry of the output, or m^2 multiplications (25) in total to compute uv^T.

1.3 Problem 3

(Similar to Strang 2.6 problem 22.)

In pset 1, you did "upside-down" Gaussian elimination to convert the matrix

$$A = \begin{pmatrix} 1 & 6 & -3 \\ -2 & 3 & 4 \\ 1 & 0 & -2 \end{pmatrix}$$

into a *lower* triangular matrix

$$L = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

Find an upper-triangular matrix U such that $A = UL$. (This example illustrates the fact that "upside-down" elimination corresponds to a "UL factorization" of A.)

1.4 Problem 4

1.4.1 part (a)

Show that by multiplying a lower-triangular L matrix by a permutation (re-ordering) matrix P on the left *and* right you can convert L to an upper-triangular matrix PLP. You don't have to prove it in general, just find the matrix P that works for *any* 3×3 matrix L.

Once you have figured it out, check it. Enter your matrix P in Julia below, and use it to flip the following lower-triangular matrix to an upper-triangular one:

```
In [ ]: X = [ 1  0  0
              2  3  0
              1  3 -1]
```

```
In [ ]: P = [ ... enter your matrix here ... ]
```

```
In [ ]: P*X*P  # the result of this should be upper-triangular:
```

1.4.2 part (b)

What is P^{-1}?

You can find it numerically from Julia with the command `inv(P)`, but you should still explain *why* it comes out that way:

```
In [ ]: inv(P) # compute P^-1 numerically
```

1.4.3 part (c)

Suppose we take the A matrix from problem 3 and the P matrix from above, and compute the LU factorization $PAP = L'U'$ without row swaps (labeling the matrices L' and U' so that they aren't confused with the ones above), then compute $PL'P$ and $PU'P$. How to the results compare to your $A = UL$ factorization from problem 3?

Why? (You should be able to do some matrix algebra to turn $PAP = L'U'$ into $A = UL$.)

```
In [ ]: A = [ 1  6 -3
             -2  3  4
              1  0 -2 ]
        L′, U′ = lu(P*A*P, Val{false}) # LU factorization of PAP without row swaps
```

```
In [ ]: P*L′*P
```

```
In [ ]: P*U′*P
```