

pset10

September 7, 2017

1 18.06 Pset 10

Due Wednesday, 5/3. (Note that 18.06 exam 3 is on Friday, 5/5; see the [course page](#) for topics.)

1.1 Problem 1

(From Strang, section 6.5, problem 15.)

Show that if $m \times m$ matrices S and T are positive-definite, then $S + T$ is positive-definite. (Use one of the tests for positive-definiteness, from lecture or from the book.)

1.2 Problem 2

In class, we showed that a line of n identical masses connected $n + 1$ springs (anchored at the two ends) leads to an ODE of the form:

$$\ddot{x} = -D^T W D x$$

where x is the vector of the n displacements of the masses, W is a diagonal matrix of the spring constants k_j divided by the masses, D is an $(n + 1) \times n$ incidence matrix (which we proved is full column rank in class):

$$D = \begin{pmatrix} 1 & & & & & & \\ -1 & 1 & & & & & \\ & -1 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & -1 & 1 & & \\ & & & & -1 & & \end{pmatrix}.$$

The fact that $A = D^T W D$ is positive-definite was crucial, because this meant that the *oscillation frequencies* $\omega = \sqrt{\lambda}$ of the vibrating “modes” of the system were *real* (hence, the masses oscillate, they don’t exponentially grow or decay).

Now, suppose we attack the *same* problem, but the masses are *not* identical. In this case, it is easy to repeat the derivation (*you don’t need to*) and show that we get equations of the form:

$$\ddot{x} = -M^{-1} D^T K D x$$

where M is the diagonal matrix of the n (positive) masses m_1, m_2, \dots, m_n and K is the diagonal matrix of the $n + 1$ (positive) spring constants k_1, k_2, \dots, k_{n+1} . This matrix $M^{-1} D^T K D$ is *not* symmetric, so at first you might be worried that you could get complex eigenvalues, and hence (physically impossible) exponentially growing or decaying solutions.

1.2.1 (a)

Show that if you do a **change of variables** $y = Sx$, where S is some **diagonal matrix**, then you get an equation $\ddot{y} = -Ay$ where A is real-symmetric positive-definite (and hence you get real oscillating frequencies $\omega = \sqrt{\lambda}$ exactly as in class).

Hint: S is a diagonal matrix involving the *square roots* of the masses.

1.2.2 (b)

Show that your matrix A is **similar** to $M^{-1}D^TKD$, so that the latter also must have real eigenvalues.

1.2.3 (c)

The following code generates 20 random masses and 21 random spring constants, and computes the eigenvalues of $M^{-1}D^TKD$.

Add another line to compute the eigenvalues of your matrix A from (a), and verify that the eigenvalues are the same. Note: to create a diagonal matrix of the square roots of the masses in Julia, you can do `diagm(sqrt.(m))`.

```
In [ ]: m = rand(20) # 20 random masses between 0 and 1
        k = rand(21) # 21 random spring constants between 0 and 1
        M = diagm(m) # diagonal matrix of the masses
        K = diagm(k) # diagonal matrix of the spring constants
        o = ones(20); D = full(spdiagm((o,-o),(0,-1),21,20)) # the 21x20 D matrix
        eigvals(M \ D' * K * D)

In [ ]: eigvals(???) # fix this
```