# pset10-sol

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# 1 18.06 Pset 10 - Solutions

## 1.1 Problem 1

(From Strang, section 6.5, problem 15.)

Show that if $m \times m$ matrices $S$ and $T$ are positive-definite, then $S+T$ is positive-definite. (Use one of the tests for positive-definiteness, from lecture or from the book.)

### 1.1.1 Solution

If $S, T$ are positive definite, then for all nonzero $x$ $x^H S x > 0$ and $x^H T x > 0$. In particular

$$x^H (S+T) x = x^H S x + x^H T x > 0$$

So $S+T$ is also positive definite.

## 1.2 Problem 2

In class, we showed that a line of $n$ identical masses connected $n+1$ springs (anchored at the two ends) leads to an ODE of the form:

$$\ddot{x} = -D^T W D x$$

where $x$ is the vector of the $n$ displacements of the masses, $W$ is a diagonal matrix of the spring constants $k_j$ divided by the masses, $D$ is an $(n+1) \times n$ incidence matrix (which we proved is full column rank in class):

$$D = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}.$$

The fact that $A = D^T W D$ is positive-definite was crucial, because this meant that the *oscillation frequencies* $\omega = \sqrt{\lambda}$ of the vibrating "modes" of the system were *real* (hence, the masses oscillate, they don't exponentially grow or decay).

Now, suppose we attack the *same* problem, but the masses are *not* identical. In this case, it is easy to repeat the derivation (*you don't need to*) and show that we get equations of the form:

$$\ddot{x} = -M^{-1} D^T K D x$$

where $M$ is the diagonal matrix of the $n$ (positive) masses $m_1, m_2, \ldots, m_n$ and $K$ is the diagonal matrix of the $n+1$ (positive) spring constants $k_1, k_2, \ldots, k_{n+1}$. This matrix $M^{-1} D^T K D$ is *not* symmetric, so at first you might be worried that you could get complex eigenvalues, and hence (physically impossible) exponentially growing or decaying solutions.

### 1.2.1 (a)

Show that if you do a **change of variables** $y = Sx$, where $S$ is some **diagonal matrix**, then you get an equation $\ddot{y} = -Ay$ where $A$ is real-symmetric positive-definite (and hence you get real oscillating frequencies $\omega = \sqrt{\lambda}$ exactly as in class).

Hint: $S$ is a diagonal matrix involving the *square roots* of the masses.

### 1.2.2 (b)

Show that your matrix $A$ is **similar** to $M^{-1}D^T KD$, so that the latter also must have real eigenvalues.

### 1.2.3 (c)

The following code generates 20 random masses and 21 random spring constants, and computes the eigenvalues of $M^{-1}D^T KD$.

Add another line to compute the eigenvalues of your matrix $A$ from (a), and verify that the eigenvalues are the same. Note: to create a diagonal matrix of the square roots of the masses in Julia, you can do `diagm(sqrt.(m))`.

In [1]:
```
m = rand(20) # 20 random masses between 0 and 1
k = rand(21) # 21 random spring constants between 0 and 1
M = diagm(m) # diagonal matrix of the masses
K = diagm(k) # diagonal matrix of the spring constants
o = ones(20); D = full(spdiagm((o,-o),(0,-1),21,20)) # the 21×20 D matrix
eigvals(M \ D' * K * D)
```

Out[1]:
```
20-element Array{Float64,1}:
 378.393
  22.7784
   7.19492
   6.99004
   4.6874
   4.36002
   4.29697
   4.0479
   2.95292
   2.09382
   1.37472
   1.23118
   0.0101793
   0.0575899
   0.232484
   0.92327
   0.838373
   0.379284
   0.515733
   0.60174
```

In [2]:
```
eigvals(???) # fix this
```

```
syntax: colon expected in "?" expression
```

### 1.2.4 Solutions

**(a)** After doing a change of variable $y = Sx$, that is $x = S^{-1}y$, the equation becomes

$$S^{-1}\ddot{y} = -M^{-1}D^TKDS^{-1}y \Leftrightarrow \ddot{y} = -SM^{-1}D^TKDS^{-1}y$$

So we need the matrix $A = SM^{-1}D^TKDS^{-1}$ to be symmetric. That is

$$SM^{-1}D^TKDS^{-1} = (SM^{-1}D^TKDS^{-1})^T = (S^T)^{-1}D^TKDM^{-1}S^T$$

So it is enough to have $M^{-1}S^T = S^{-1}$ or, equivalently, $M = S^TS$. Since $M$ is a diagonal matrix with positive diagonal entries it is enough to choose $S$ diagonal where the diagonal entries of $S$ are the square roots of the diagonal entries of $M$ (that is the masses). Then, substituting $M = S^TS$ we get

$$A = SM^{-1}D^TKDS^{-1} = SS^{-1}(S^{-1})^TD^TKDS^{-1} = (DS^{-1})^TK(DS^{-1})$$

which is clearly symmetric. It only remains to check that it is definite positive. Now let $v$ be a nonzero real vector, we need to check $v^TAv > 0$. But

$$v^TAv = v^T(DS^{-1})^TK(DS^{-1})v = (DS^{-1}v)^TK(DS^{-1}v) > 0$$

since $DS^{-1}v$ is a nonzero vector and $K$ is definite positive. #### (b) Our matrix $A$ was defined as

$$A = S(M^{-1}D^TKD)S^{-1}$$

and it is plainly similar to $M^{-1}D^TKD$ (with similarity matrix $S^{-1}$).

**(c)** By using the formula $A = S^{-1}D^TKDS^{-1}$ we have

In [3]:
```
S=diagm(sqrt.(m))
eigvals(inv(S) *  D' * K * D * inv(S))
```

Out[3]:
```
20-element Array{Float64,1}:
 378.393
  22.7784
   7.19492
   6.99004
   4.6874
   4.0479
   4.36002
   4.29697
   2.95292
   2.09382
   1.37472
   1.23118
   0.0101793
   0.0575899
   0.232484
   0.92327
   0.838373
   0.379284
   0.515733
   0.60174
```

which coincides with the answer for $M^{-1}D^TKD$.

In [ ]: