MIT 18.06 Final Exam Solutions, Spring 2017

Problem 1:

For some real matrix A, the following vectors form a basis for its column space and null space:

$$C(A) = \text{span}\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \rangle,$$

$$N(A) = \text{span}\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \rangle.$$

(a) What is the size $m \times n$ of A, what is its rank, and what are the dimensions of $C(A^T)$ and $N(A^T)$?

Solution: A must be a 3×5 matrix of rank 2 (the dimension of the column space). $C(A^T)$ must have the same dimension 2, and $N(A^T)$ must have dimension $3 - 2 = 1$.

(b) Give *one* possible matrix A with this $C(A)$ and $N(A)$.

Solution: We have to make all the columns out of the two $C(A)$ vectors. Let's make the first column $(1, 0, 1)$. From the first nullspace vector, the second column must then be $(1, 0, 1)$; from the second nullspace vector, the fifth column must be $(-2, 0, -2)$; from the third nullspace vector, the third column must be $(3, 0, 3)$. The fourth column must be independent and give us the other $C(A)$ vector, so we can just make it $(1, 1, -1)$. Hence, our A matrix is

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & -1 & -2 \end{pmatrix}.$$

Of course, there are many other possible solutions.

(c) Give a right-hand side b for which $Ax = b$ has a solution, and give *all* the solutions x for your A from the previous part. (Hint: you should not have to do Gaussian elimination.)

Solution: we just need b to be in the column space, e.g. $b = (1, 0, 1)$. Then a particular solution is $(1, 0, 0, 0, 0)$, and to get all possible solutions we just need to add any multiples of the nullspace vectors:

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

for any scalars c_1, c_2, c_3. (Again, there are many possible solutions to this part.)

(d) For $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, the equation $Ax = b$ as *no solutions*. Instead, give another right-hand side $\hat{b}$ for which $A\hat{x} = \hat{b}$ is solvable and yields a least-square solution $\hat{x}$ (i.e. $\hat{x}$ minimizes $\|Ax - b\|$). $\hat{b}$ must be the ____________________________ of b onto the subspace ____________________. (Hint: if you find yourself solving a 4×4 system of equations, you are missing a way to do it much more easily. The answer should *not* depend on your choice of A matrix in part b.)

Solution: We just need to project b onto $C(A)$. If you look closely, you'll notice that the basis of $C(A)$ given above is actually orthogonal (but not orthonormal), so the orthogonal projection is easy. If we call the two basis vectors a_1 and a_2, then the projection is

$$\hat{b} = \frac{a_1 a_1^T}{a_1^T a_1} b + \frac{a_2 a_2^T}{a_2^T a_2} b = a_1 \frac{1}{2} + a_2 \frac{1}{3} = \begin{pmatrix} 5/6 \\ 1/3 \\ 1/6 \end{pmatrix}.$$

We could also have used the projection formula Pb for $P = \hat{A}(\hat{A}^T \hat{A})^{-1} \hat{A}^T$ where $\hat{A} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$. This is pretty easy, too, since $\hat{A}^T \hat{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, making it easy to invert (it is diagonal because the basis is orthogonal).

Problem 2:

Suppose you have 100 data points (x_i, y_i) for $i = 1, 2, \ldots, 100$, and you want to fit them to a power-law curve $y(x) = ax^b$ for some a and b. Equivalently, you want to fit $\log y_i$ to $\log y = \log(ax^b) = b \log x + \log a$. Describe how to find a and b to *minimize the sum of the squares of the errors:*

$$s(a,b) = \sum_{i=1}^{100} \left(b \log x_i + \log a - \log y_i\right)^2 .$$

Write down a 2×2 system of equations for the vector $z = \begin{pmatrix} b \\ \log a \end{pmatrix}$; you can leave your equations in the form of a product of matrices/vectors as long as you say what the matrices/vectors are. (Hint: rewrite it as an 18.06-style least-squares problem with matrices/vectors.)

Solution: In linear-algebra form, we write $s = \|Az - b\|^2$ where A is the 100×2 matrix

$$A = \begin{pmatrix} \log x_1 & 1 \\ \log x_2 & 1 \\ \vdots & \vdots \\ \log x_{100} & 1 \end{pmatrix}$$

and b is the 100-component vector

$$b = \begin{pmatrix} \log y_1 \\ \log y_2 \\ \vdots \\ \log y_{100} \end{pmatrix}.$$

Then minimizing s over z is just an ordinary least-squares problem, and the solution is given by the normal equations $A^T A z = A^T b$, which is a 2×2 system of equations.

Problem 3:

Suppose that 4×4 real matrix $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}$ has four *orthogonal* but *not orthonormal* columns a_i with lengths $\|a_1\| = 2$, $\|a_2\| = 1$, $\|a_3\| = 3$, $\|a_4\| = 2$. (That is, $a_i^T a_j = 0$ for $i \neq j$.)

(a) Write an explicit expression for the solution x to $Ax = b$ in terms of dot products, additions, and multiplications by scalars.

Solution: All we are doing is writing $b = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$, i.e. we are writing b in the basis of the columns of A. Since this is an orthogonal basis, we have seen many times in class that we just need to take dot products: $a_i^T b = a_i^T a_i x_i$ for $i = 1, 2, 3, 4$, so $x_i = a_i^T b / a_i^T a_i$ and

$$x = \begin{pmatrix} a_1^T b / a_1^T a_1 \\ a_2^T b / a_2^T a_2 \\ a_3^T b / a_3^T a_3 \\ a_4^T b / a_4^T a_4 \end{pmatrix} = \begin{pmatrix} a_1^T b / 4 \\ a_2^T b \\ a_3^T b / 9 \\ a_4^T b / 4 \end{pmatrix}.$$

Equivalently, since $A^T A$ is the diagonal matrix $D = \begin{pmatrix} a_1^T a_1 & & & \\ & a_2^T a_2 & & \\ & & a_3^T a_3 & \\ & & & a_4^T a_4 \end{pmatrix} = \begin{pmatrix} 4 & & & \\ & 1 & & \\ & & 9 & \\ & & & 4 \end{pmatrix}$, we have $D^{-1}A^T A = I$, which means that $A^{-1} = D^{-1}A^T$, and hence $x = D^{-1}A^T b$. If you write it out, this is essentially the same as the answer above (note that inverting a diagonal matrix is easy).

(b) Write A as a sum of four rank-1 matrices.

Solution: If you understand how rank-1 matrices (outer products) work, there is a trivial way to do this. If we let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, then

$$A = a_1 e_1^T + a_2 e_2^T + a_3 e_3^T + a_4 e_4^T.$$

(c) If we write the matrix $B = A \begin{pmatrix} 3 & & & \\ & 6 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$, then what is $B^T B$? Hence, for any $x \neq 0$, $\frac{\|Bx\|}{\|x\|} =$ ______.

Solution: If $B = AS$ where S is the diagonal matrix above, then $B^T B = S^T A^T A S = SDS$ where D is the diagonal matrix above, and multiplying out SDS gives

$$B^T B = \begin{pmatrix} 4 \times 3^2 & & & \\ & 6^2 & & \\ & & 9 \times 2^2 & \\ & & & 4 \times 3^2 \end{pmatrix} = 36I.$$

(That is, $B/6$ is actually a unitary matrix.) Hence $\|Bx\| = \sqrt{(Bx)^H(Bx)} = \sqrt{x^H B^T B x} = \sqrt{36 x^H x} = 6\|x\|$, and $\frac{\|Bx\|}{\|x\|} = 6$.

(d) Write the SVD $A = U\Sigma V^T$: explicitly give the singular values σ (diagonal of Σ) and the singular vectors (columns of U, V, possibly in terms of the columns a_i). Hint: what is $A^T A$, and what are its eigenvectors (this should give you either U or V) and eigenvalues (related somehow to σ)? Recall also from homework that $AV = U\Sigma$.

Solution:

$$\Sigma = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 3 & \\ & & & 2 \end{pmatrix}$$

with

$$U = \begin{pmatrix} \frac{a_1}{2} & \frac{a_2}{1} & \frac{a_3}{3} & \frac{a_4}{2} \end{pmatrix}$$

and $V = I$. There are many ways to see this. The easiest way is by inspection if you really understand the SVD: the columns of A are already orthogonal, so we just need to normalize them to length 1 to get an orthonormal basis U of the column space, where we recover A just by multiplying by the diagonal matrix Σ of the lengths. But, if we want to do it the "long way," it is not too bad either.

The "long way" is to first find the eigenvalues and eigenvectors of $A^T A = V\Sigma^T \Sigma V^T$. The nonzero eigenvalues are the σ_i^2 , and the corresponding eigenvectors are the columns of V . But this is easy, since $A^T A$ is diagonal! Hence we see that the singular values σ are just the lengths $2, 1, 3, 2$ of the four columns of A. The eigenvectors of a diagonal matrix are just $V = I$. Then we get U from $AV\Sigma^{-1}$: that is (as we saw in homework), for each right singular vector v_i and nonzero singular value σ_i, there is a corresponding left singular vector $u_i = Av_i/\sigma_i$. But since $V = I$ and the σ_i are just the lengths of the four columns of A, we immediately see that U consists of the columns of A normalized by their lengths.

Problem 4:

Suppose that A and B are two real $m \times m$ matrices, and B is invertible.

(a) Circle which (if any) of the following must be true: $C(A) = C(AB)$, $C(A) = C(BA)$, $N(A) = N(AB)$, $N(A) = N(BA)$.

Solution: $C(A) = C(AB)$ and $N(A) = N(BA)$.

Proof (not required): Since B is invertible, if $y \in C(A)$ then $y = Ax$ for some x and $y = ABz$ for $z = B^{-1}x$, hence $y \in C(AB)$. Similarly, if $Ax = 0$ [$x \in N(A)$] then $BAx = 0$ [$x \in N(BA)$] and vice versa (multiplying both sides by B^{-1}). However, $C(A) \neq C(BA)$: a counterexample is $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for which $C(A)$ is spanned by $(1, 0)$ and $C(BA)$ is spanned by $(0, 1)$. Similarly, $N(A) \neq N(AB)$ for the same counterxample: $N(A)$ is spanned by $(0, 1)$ and $N(AB)$ is spanned by $(1, 0)$.

(b) Circle which (if any) of the following matrices must be symmetric: $ABBA$, A^TBB^TA, A^TBAB^T, $A^TBB^{-1}A$, $A^T(B^T)^{-1}B^{-1}A$?

Solution: A^TBB^TA, $A^TBB^{-1}A = A^TA$, and $A^T(B^T)^{-1}B^{-1}A = A^T(B^{-1})^TB^{-1}A$ are symmetric, as can easily be seen by applying the transpose formula (the transpose of the product is the product of the transposes in reverse order).

(c) If B is a projection matrix (as well as invertible), then $AB =$ ___________.

Solution: An invertible projection matrix must be $B = I$, hence $AB = A$.

(d) Do AB and BA have the same eigenvalues? Give a reason if true, a counter-example if false.

Solution: They are similar ($AB = B^{-1}BAB$), and hence have the same eigenvalues.

(e) Suppose A has rank r. Say as many true things as possible about the eigenvalues of $C = A^TB^TBA$ that would *not* necessarily be true if C were just a random $m \times m$ matrix.

It is real-symmetric, so the eigenvalues of C are real. $N(A^TB^TBA) = N(A)$ so C has exactly $m - r$ zero eigenvalues, and it is positive semi-definite so the remaining r eigenvalues are positive.

Problem 5:

You have a matrix A with the factorization:

$$A = \underbrace{\begin{pmatrix} 1 & & \\ 3 & 2 & \\ 1 & -1 & 2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 1 & 3 & 1 \\ & 2 & -1 \\ & & 2 \end{pmatrix}}_{C=B^T}.$$

(a) What is the product of the 3 eigenvalues of A?

Solution: The product of the eigenvalues is $\det A = \det B \times \det C$. B and C are triangular matrices so their determinants are just the products of their diagonals $= 1 \times 2 \times 2 = 4$, hence $\det A = 4^2 = 16$.

(b) Solve $Ax = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$ for x. (Hint: if you find yourself doing Gaussian elimination, you are missing something.)

Solution: The purpose of Gaussian elimination is to factorize A into a product of triangular matrices, but this A is *already* factorized into a product of triangular matrices. So, we just need to do one back-substitution and one forward-substitution step. In particular, since $Ax = b = BCx$, we let $Cx = y$ and first solve $By = b$ for y by forward substitution (since B is upper triangular) and then solve $Cx = y$ for x by backsubstitution. To solve $By = b$:

$$\begin{pmatrix} 1 & & \\ 3 & 2 & \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} \implies y_1 = 2 \implies \begin{array}{c} 3y_1 + 2y_2 = 4 \\ \implies y_2 = -1 \end{array} \implies \begin{array}{c} y_1 - y_2 + 2y_3 = 7 \\ \implies y_3 = 2 \end{array}.$$

Now that we know $y = (2, -1, 2)$, we solve $Cx = y$ by backsubstitution:

$$\begin{pmatrix} 1 & 3 & 1 \\ & 2 & -1 \\ & & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \implies x_3 = 1 \implies \begin{array}{c} 2x_2 - x_3 = -1 \\ \implies x_2 = 0 \end{array} \implies \begin{array}{c} x_1 + 3x_2 + x_3 = 2 \\ \implies x_1 = 1 \end{array}$$

and hence the solution is $x = (1, 0, 1)$.

There are alternative methods. You could multiply out BC to find $A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 13 & 1 \\ 1 & 1 & 6 \end{pmatrix}$. If you stare at this matrix for a little while, you might notice that $(2, 4, 7)$ is the sum of the first and third columns, which immediately gives the solution (this often works in the toy problems of 18.06 where the solutions are almost always small integers). You could, of course, do Gaussian elimination on A; this works, but is a lot of pointless work because triangular factors of A were already given to you! You could also write $x = A^{-1}b = C^{-1}B^{-1}b$, but if you find yourself explicitly computing inverse matrices you are almost always doing more work than you should — you should read an expression like "$B^{-1}b$" as "solve $By = b$" (usually by elimination, but in this case by forward-substitution).

(c) Gaussian elimination (without row swaps) produces an $A = LU$ factorization, but you can tell at a glance that this is *not* the same as the factorization above, because L is always a lower-triangular matrix with 1's on the diagonal. Find the ordinary LU factorization of A (the matrices L and U) by multiplying and/or dividing the factors above with some diagonal matrix.

Solution: We just write $A = BC = BD^{-1}DC$ where D is the diagonal matrix $\begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix}$. This divides the columns of B by the diagonal elements to make them equal to 1:

$$A = \underbrace{\begin{pmatrix} 1 & & \\ 3 & 2 & \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ & 2 & -1 \\ & & 2 \end{pmatrix}}_{U} = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ 1 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ & 4 & -2 \\ & & 4 \end{pmatrix}.$$

Problem 6:

(a) One of of the eigenvalues of

$$A = \begin{pmatrix} 3 & 1 & 4 \\ & 1 & 5 \\ & 1 & 5 \end{pmatrix}.$$

is $\lambda = 3$. Find the other two eigenvalues. (Hint: one eigenvalue should be obvious from inspection of A, since A is ________________. You shouldn't need to explicitly write down and solve any cubic equation, because once you find two eigenvalues you can get the third from the ________________ of A.)

Solution: A is obviously singular (the second and third rows are the same), so it must have an eigenvalue $\lambda = 0$. The trace is $3 + 1 + 5 = 9$ which must be the sum of the eigenvalues, so the third eigenvalue must be $\lambda = 9 - 3 - 0 = 6$.

(b) For a 4×4 matrix B, the polynomial $\det(B - \lambda I)$ has three roots $\lambda = 1, 0.4, -0.7$. You find that, for some vector x, the vector $B^n x$ is getting longer and longer as n grows. It must be the case that B is a __________________________ matrix. Approximately what is $\|B^{2000}x\|/\|B^{1000}x\|$?

Solution: The matrix B must be defective. (If it were diagonalizable, then $B^n x$ would only contain λ^n terms, which don't grow since $|\lambda| \leq 1$ here.) It is 4×4 with 3 distinct eigenvalues, so one of the eigenvalues must be a double root, corresponding to a 2×2 Jordan block and a single additional Jordan vector to supplement the three eigenvectors. For such a defective matrix, there is a term (from the Jordan vector) that goes as $n\lambda^n$. For this to grow, it must be the $\lambda = 1$ eigenvalue that is repeated, giving a term that grows proportional to n. This must dominate for large n, hence we should have $\|B^{2000}x\|/\|B^{1000}x\| \approx 2$.

More explicitly, call the three eigenvectors x_1, x_2, x_3, and the Jordan vector j_1. If we write $x = c_1 x_1 + d_1 j_1 + c_2 x_2 + c_3 x_3$ in this basis for some coefficients c_1, d_1, c_2, c_3, then we will have:

$$B^n x = 1^n c_1 x_1 + 1^n d_1 j_1 + n 1^{n-1} d_1 x_1 + 0.4^n c_2 x_2 + (-0.7)^n c_3 x_3,$$

and for large n this is $\approx n d_1 x_1$.

(c) A positive Markov matrix M has a steady-state eigenvector $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}$. What is $M^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ for $n \to \infty$?

Solution: $M^n x$ must approach a multiple of the steady-state eigenvector for any x . Which multiple? Multiplying by a Markov matrix conserves the sum of the entries of a vector, and the sum of the entries of $(1, 0, 0, 0)$ is 1, so it must approach

$$\frac{1}{6}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}.$$

(d) For a real matrix C, and almost any randomly chosen initial $x(0)$, the equation $\frac{dx}{dt} = Cx$ has solutions that are *oscillating and decaying* (for large t) as a function of t. **Circle all of the things** that could **possibly be true** of C: symmetric, anti-symmetric, orthogonal, Markov, diagonalizable, defective, singular, diagonal.

Solution: to have solutions that are oscillating and decaying, there must be a *complex* eigenvalue λ with a *negative real part*. Furthermore, to get this for "almost any" initial condition, we must have that the *other* solutions are decaying (faster) too: the other eigenvalues must have (larger) negative real parts. This immediately rules out symmetric (real λ), antisymmetric (imaginary λ), Markov (a $\lambda = 1$), and singular (a $\lambda = 0$) matrices. It can't be diagonal because the matrix is real, and a real diagonal matrix has real eigenvalues (the diagonal entries). This leaves: **orthogonal, diagonalizable,** or **defective**. All of those are possible. (Orthogonal matrices can have complex eigenvalues with negative real parts, as long as $|\lambda| = 1$. Both diagonalizable and defective matrices can have such eigenvalues, of course. Defective matrices give a term in $x(t)$ that goes like $te^{\lambda t}$, but this is still decaying if λ has a negative real part.)

Problem 7:

The matrix A is real-symmetric and positive-definite. Using it, we write a recurrence equation

$$x_n - x_{n+1} = A\,(x_n + x_{n+1})$$

for a sequence of vectors $x_0, x_1, \ldots$

(a) For any x_n, the recurrence relation above defines a unique solution x_{n+1}. Why?

Solution: First, let's rearrange the recurrence to get an equation for x_{n+1}. We have $x_n - Ax_n = x_{n+1} + Ax_{n+1}$ or, in matrix form:

$$(I + A)x_{n+1} = (I - A)x_n \implies x_{n+1} = (I + A)^{-1}(I - A)x_n.$$

This always has a unique solution since $I + A$ is invertible: A is positive-definite with eigenvalues $\lambda > 0$, so $I + A$ has eigenvalues $1 + \lambda > 0$.

(b) If $Av = \lambda v$ and $x_0 = v$, give an equation for x_n in terms of v, n, and λ.

Solution: From above,

$$x_n = \left[(I + A)^{-1}(I - A)\right]^n x_0.$$

If x_0 is an eigenvector v of A (also an eigenvector of $I + A$ and $I - A$), then A just acts like a scalar λ and we get

$$x_n = \left[\frac{1-\lambda}{1+\lambda}\right]^n v$$

(c) For an arbitrary x_0, does the length of the solution $\|x_n\|$ grow with n, decay with n, oscillate, or approach a nonzero steady-state?

Solution: We can expand any vector x_0 in the basis of the eigenvectors $v_1, v_2, \ldots, v_n$, i.e. $x_0 = \sum_i c_i v_i$. Each term uses the formula from the previous part, so

$$x_n = \sum_i \left[\frac{1-\lambda_i}{1+\lambda_i}\right]^n c_i v_i.$$

But **each of these terms is decaying**: $\lambda_i > 0$ (A is positive-definite), so $|1 - \lambda| < |1 + \lambda|$ and the ratio has $\left|\frac{1-\lambda_i}{1+\lambda_i}\right| < 1$. So, $\|x_n\| \to 0$ as $n \to \infty$.

(d) Suppose A is 4×4 and the eigenvalues are $\lambda_1 = 0.1$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 3$, and the corresponding eigenvectors are v_1, v_2, v_3, and v_4 (all normalized to length $\|v_k\| = 1$). The initial x_0 is some arbitrary vector. Write an exact formula for x_n in terms of x_0 and these eigenvectors and eigenvalues—you should get a sum of four terms. Which term should typically dominate for large n?

Solution: As above, but we can furthermore use the fact that the eigenvectors are orthonormal (they are orthogonal since it is a real-symmetric matrix with distinct eigenvalues, and we normalized them to length 1) to say that the coefficients are $c_i = v_i^T x_0$, and hence

$$x_n = \sum_{i=1}^{4} \left[\frac{1-\lambda_i}{1+\lambda_i}\right]^n v_i v_i^T x_0 = \left[\frac{0.9}{1.1}\right]^n v_1 v_1^T x_0 + 0^n v_2 v_2^T x_0 + \left[-\frac{1}{3}\right]^n v_3 v_3^T x_0 + \left[-\frac{1}{2}\right]^n v_4 v_4^T x_0.$$

For large n, the first term obviously dominates.