MIT 18.06 Exam 3 Solutions, Spring 2017

Problem 1:

The following matrix M is a Markov matrix (its columns sum to 1):

$$M = \begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix}$$

and its steady-state eigenvector ($\lambda = 1$) is

$$s = \begin{pmatrix} 7/18 \\ 1/3 \\ 5/18 \end{pmatrix}.$$

Recall from class that multiplying a vector x by M does not change the sum of the components. That is, the sum $= o^T x$, where $o = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, is conserved:

$$o^T M x = o^T x = x_1 + x_2 + x_3$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. (The steady-state vector s above was normalized so that $o^T s = 1$.)

(a) If we let $x = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, what vector does $M^n x$ approach as $n \to \infty$?

Solution: Since this is a positive Markov matrix, all other eigenvalues have magnitude < 1, and the solution must approach a multiple of s. Which multiple? Well, the sum of the entries is conserved, so the sum must equal $o^T x = 2$. Hence $\boxed{M^n x \to 2s}$.

(b) For the same x, in what direction does $(M^T)^n x$ point as $n \to \infty$. (You don't have to give the correct *magnitude*, just give a vector in the correct *direction*.)

Solution: Similar to the previous part, except that the steady state eigenvector is o, since $M^T o = o$ (this is just the statement that the columns of M sum to 1, written in matrix form). Hence $\boxed{(M^T)^n x \to \alpha o}$ for some scalar coefficient α. (From the solution of the next part, we must have $s^T(\alpha o) = s^T x$, and we can compute $\boxed{\alpha = \frac{s^T x}{s^T o} = 7/9}$. You *weren't required* to do this, however.)

(c) Multiplying $M^T x$ does *not* conserve the *sum* of the components of x, unlike Mx. However, it does conserve *some* linear combination of the components: there is some vector $v \neq 0$ such that

$$v^T M^T x = v^T x$$

for all x. What is v? (Hint: this is easy if you understand *why* $o^T M x = o^T x$ as stated above.)

Solution: The reason why $o^T M x = o^T x$ was simply that o is an eigenvector of M^T with eigenvalue $\lambda = 1$ (a "left eigenvector" of M). Here, we need a similar eigenvector of M, and we have one: $\boxed{v = s}$ (or any multiple of s), since $s^T M^T x = (Ms)^T x = s^T x$.

(Erratum: the problem originally failed to specify $v \neq 0$, in which case $v = 0$ is a valid, if trivial, solution.)

Problem 2:

Suppose that A is a 3×3 real-symmetric matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -2$, and corresponding eigenvectors x_1, x_2, x_3. You are given that $x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(a) Give an approximate solution at $t = 100$ to $\frac{dx}{dt} = Ax$ for $x(0) = (1, 1, 0)$. (You should give a specific vector, even if the vector is very big or very small — an answer of "≈ 0" or "$\approx \infty$" is not acceptable.)

Solution: If we write $x(t) = c_1 x_1 e^t + c_2 x_2 e^{-t} + c_3 x_3 e^{-2t}$, then for $t = 100$ it is clear that the c_1 term dominates. Furthermore, since A is real-symmetric, the eigenvectors must be orthogonal, and hence $1 = x_1^T x(0) = c_1 x_1^T x_1 = 2c_1$, or $c_1 = 0.5$. Hence $\boxed{x(100) \approx 0.5 e^{100} x_1}$.

(b) If $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, what is x_3? (You should *not* need your answer here to solve the previous part!)

Solution: Since x_3 must be orthogonal to x_1 and x_2 (A is real-symmetric with distinct λ's), the only possibility is $\boxed{x_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$ or some nonzero multiple thereof.

(c) If instead we solve $\frac{dx}{dt} = (\alpha I - A^3)x$ for some *complex* number $\alpha$and the same $x(0)$, give a possible value of α for which the solutions $x(t)$ approach *oscillating* solutions (but not decaying or growing!) for large t.

Solution: The eigenvalues of $\alpha I - A^3$ are $\alpha - \lambda_k^3$, or $\alpha - 1$, $\alpha + 1$, and $\alpha + 8$ (with the same eigenvectors). To have oscillating solutions at a large t, one of these eigenvalues must be purely imaginary, and the other eigenvalues must have negative real parts. So, we must have $\mathrm{Re}(\alpha) = -8$ (to cancel the real part of the largest term), and some imaginary part (any imaginary part we want). Hence, the allowed solutions are $\boxed{\alpha = -8 + i\omega}$ for any real $\omega \neq 0$ (e.g. $\omega = 1$ is fine).

Problem 3:

The real 3×3 matrix A is positive-definite, and the real 3×4 matrix B is rank 3:

$$B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The nullspace $N(B)$ is spanned by the vector $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

(a) How many zero, positive, and negative eigenvalues does B^TAB have? (Hint: what happens if you plug an eigenvector into $x^T(B^TAB)x$?)

Solution: If $B^TABx = \lambda x$, then $x^TB^TABx = \lambda x^Tx = y^TAy$ where $y = Bx$. $x^Tx > 0$, and because A is positive-definite we know that $y^TAy \geq 0$, so it immediately follows that $\lambda \geq 0$. Furthermore, $y^TAy = 0$ only if $y = 0$, i.e. $x \in N(B)$. Since $N(B)$ is one-dimensional, this means that there is only **one zero eigenvalue** (with eigenvector x_0) and the **remaining three eigenvalues are positive**. (There are four eigenvalues because B^TAB is a 4×4 matrix. Of course, it is possible that some of the positive three eigenvalues are repeated, e.g. if $A = I$.)

In fact, B^TAB is positive semidefinite for any real B and any positive-definite A, and nullspace is the same as B.

(b) For which sign (+ or −) does $\frac{dx}{dt} = \pm B^T ABx$ have solutions that approach a constant steady state for any initial condition $x(0)$?

Solution: −. This way, the positive eigenvalues from above give decaying solutions, and the zero eigenvalue gives a steady state.

(c) For the sign you chose in the previous part, what is $x(\infty)$ for $x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$?

Solution: Since $B^T AB$ is real-symmetric, the eigenvalues are orthogonal, and we can get the steady-state ($\lambda = 0$) component (given by the null-space vector x_0 given above) just by a dot product (the projection of $x(0)$ onto x_0):

$$\boxed{x(\infty) = \frac{x_0 x_0^T x(0)}{x_0^T x_0} = x_0 \frac{1}{4} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.}$$

Problem 4:

True or false. Give a reason if true (one sentence and/or one equation should suffice), or a counterexample if false.

(a) A singular matrix A cannot be similar to a non-singular matrix B.

True. Similar matrices have the same eigenvalues, but B must have a zero eigenvalue and A must have nonzero eigenvalues.

(b) Any positive markov matrix M (that is, positive entries) must also be positive definite.

False. There are many ways to construct a counterexample without doing a lot of calculations. Every positive-definite matrix by definition must be Hermitian, so it is sufficient to give a non-symmetric Markov matrix, e.g. the one from problem 1. Even if the Markov matrix is real-symmetric, it can still have negative eigenvalues with magnitude < 1. For example, start with the Markov matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has eigenvalues ± 1, then add $0.5I$ to it to make a positive matrix $\begin{pmatrix} 0.5 & 1 \\ 1 & 0.5 \end{pmatrix}$ with eigenvalues -0.5 and 1.5, then divide by 1.5 (the sum of the columns) to make it a positive Markov matrix $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ with eigenvalues $-\frac{1}{3}$ and 1.

(c) If $A = QR$ is the QR factorization of a real (square) matrix A, then the matrix RQ has the same eigenvalues as A.

True. $A = QR \implies R = Q^{-1}A = Q^T A \implies RQ = Q^{-1}AQ$, which is similar to A.

(Clarification: The problem did not originally specify that A was square, but this is automatically implied by the statement that A has eigenvalues, which are only defined for square matrices.)

(d) A and e^{A^3} have the same eigenvalues.

False. If $Ax = \lambda x$, then $e^{A^3}x = e^{\lambda^3}x$, and $e^{\lambda^3} \neq \lambda$ in general. For example, consider the 1×1 matrix $A = 0$ with a single eigenvalue $\lambda = 0$, then $e^{A^3} = e^0 = 1$ has only the eigenvalue $1 \neq 0$.

(e) A and e^{A^3} have the same eigenvectors.

True. $Ax = \lambda x$, then $e^{A^3}x = e^{\lambda^3}x$, so x is also an eigenvector of e^{A^3}. (The converse also works.)