18.06 PSet 7 Solution

Problem 1

Solution: Consider

$$\cos(\alpha \pm \beta) + i\sin(\alpha \pm \beta) = e^{i(\alpha \pm \beta)} = e^{i\alpha}e^{\pm i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta \pm i\sin\beta).$$

Compare the real and imaginary parts of both sides, we get (a) and (b). In particular, take "+" sign and $\alpha = \beta$, we have

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

So we get (c) and (d). ♢

Problem 2

Solution: These identities come from direct calculation. Let us solve (d), (e) and (h) here.

Write $z = a + bi$ and $w = c + di$, then $zw = (ac - bd) + (ad + bc)i$. Therefore

$$M_z M_w = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix} = M_{zw}.$$

In particular, take $w = z^{-1}$ in the above equality, we get

$$M_z M_{z^{-1}} = M_{zz^{-1}} = M_1 = I,$$

hence

$$M_{z^{-1}} = (M_z)^{-1}.$$

Direct computation shows that

$$p_{M_z}(t) = \det \begin{pmatrix} t - a & -b \\ b & t - a \end{pmatrix} = t^2 - 2at + a^2 + b^2 = t^2 - (z + \bar{z})t + z\bar{z}.$$

♢

Problem 3

Solution: The characteristic polynomial is

$$p_A(t) = \det \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & -1 \\ -1 & \lambda & 0 & \dots & 0 & 0 \\ 0 & -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & 0 & \dots & -1 & \lambda \end{pmatrix} = \lambda^n - 1.$$

Therefore A has n distinct eigenvalues $\lambda_k = e^{2\pi(k-1)i/n}$ for $k = 1, 2, \dots, n$. These eigenvalues are all real if and only if $n = 1$ or $n = 2$. Therefore A is always diagonalizable over $\mathbb{C}$, it is diagonalizable over $\mathbb{R}$ if and only if $n = 1$ or $n = 2$. ♢

Problem 4

Solution: By Problem 1, this matrix is $M_{e^{i\theta}}$ and its characteristic polynomial is

$$t^2 - (e^{i\theta} + e^{-i\theta})t + e^{i\theta}e^{-i\theta} = (t - e^{i\theta})(t - e^{-i\theta}).$$

Therefore its eigenvalues are $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$. The corresponding eigenvectors are $(i, 1)^T$ and $(1, i)^T$ respectively. ♢

Problem 5

Solution: This is the complex version of Householder matrix (see Problem 7 of PSet 6). The only eigenvalues are 1 (multiplicity $n-1$) amd -1 (multiplicity 1). The eigenspace of eigenvalue -1 is the line spanned by $\hat{x}$ and the eigenspace of eigenvalue 1 is the hyperplane ($(n-1)$-dimensional) consisting of vectors orthogonal to $\hat{x}$, i.e. all the vectors v such that $v^*\hat{x} = 0$. ♢

Problem 6

Solution: Let the characteristic polynomial of A^{-1} be

$$\det(tI - A^{-1}) = t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0.$$

Notice that A is invertible, so $a_0 = (-1)^n \det A \neq 0$. On one hand, we have

$$\det((tI - A^{-1})A) = \det(tI - A^{-1}) \det A = (-1)^n a_0 (t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0).$$

On the other hand,

$$\begin{aligned}
\det((tI - A^{-1})A) = \det(tA - I) &= \det\left(-t\left(\frac{1}{t}I - A\right)\right) = (-t)^n \det\left(\frac{1}{t}I - A\right) \\
&= (-t)^n \left(\left(\frac{1}{t}\right)^n + a_{n-1}\left(\frac{1}{t}\right)^{n-1} + \cdots + a_0\right) \\
&= (-1)^n (a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + 1).
\end{aligned}$$

Compare these two equations, we get

$$\det(tI - A^{-1}) = t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0 = t^n + \frac{a_1}{a_0}t^{n-1} + \cdots + \frac{a_{n-1}}{a_0}t + \frac{1}{a_0}.$$

♢

Problem 7

Solution: For any complex number $\lambda \in \mathbb{C}$, the $n \times n$ matrix

$$J_\lambda := \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

is not diagonalizable over $\mathbb{C}$. This is because that though the only eigenvalue of J_λ is λ with multiplicity $n \geq 2$, its eigenspace is 1-dimensional (see Problem 6 of PSet 6). ♢