PROBLEM SET V

DUE THURSDAY, 14 APRIL 2016

(1) Suppose A a nonzero $n \times n$ matrix such that for some $k \geq 1$, the matrix $I - A^k$ is invertible. Prove that $I - A$ is invertible.

(2) Compute

$$\det\begin{pmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \\ p & q & r & s & t \\ v & w & x & y & z \end{pmatrix}$$

for any $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z \in \mathbf{R}$.

(3) Compute

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

(4) Suppose $A = (a_{i,j})_{1\le i,j\le n}$ the $n \times n$ matrix such that

$$a_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 1 & \text{if } i = j+1; \\ -1 & \text{if } i = j-1; \\ 0 & \text{otherwise.} \end{cases}$$

Let d_n be the determinant of A and let f_n be n-th the Fibonacci number. Show that $d_n = f_n$ by proving d_n satisfies the same recursion relation $d_n = d_{n-1} + d_{n-2}$ as Fibonacci numbers.

(5) (This one's a tough one!) Suppose $n \geq 2$.

(a) Fix vectors $v_1, v_2, \ldots, v_{n-1} \in \mathbf{R}^n$. Show that there exists a unique vector $x \in \mathbf{R}^n$ such that for any $w \in \mathbf{R}^n$, one has

$$x \cdot w = \det(v_1, v_2, \ldots, v_{n-1}, w).$$

In this situation, we call x the *cross product* of $v_1, v_2, \ldots, v_{n-1}$, and we write

$$x = v_1 \times v_2 \times \cdots \times v_{n-1}.$$

(Note that the cross product is a map $(\mathbf{R}^n)^{n-1} \longrightarrow \mathbf{R}^n$; it does *not* make sense to speak of the cross product of fewer than $n-1$ vectors!)

(b) Show that, for any vectors $v_1, v_2, \dots, v_{n-1} \in \mathbf{R}^n$, one has

$$|v_1 \times v_2 \times \cdots \times v_{n-1}| = \sqrt{\det M},$$

where M is the $(n-1) \times (n-1)$ matrix whose (i, j)-th entry is $v_i \cdot v_j$.