18.06.25: Similarity and diagonalizability

Lecturer: Barwick

Ambiguity is the haven of the indolent.

Let's compute the eigenvalues and eigenspaces of the following matrices.

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

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In this case, we have a repeated eigenvalue, 2. So the eigenspace is the kernel of

$$L_1 = \ker \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

which we note with a grimace is only 1-dimensional: $L_1 = \langle \hat{e}_1 \rangle$.

We have disproved our conjecture. *It is not true that the multiplicity of a root equals the dimension of the eigenspace.*

$$B = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We have three eigenvalues for $B = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$: 5, 1, and 2. We have

$$\begin{aligned} L_5 &= \langle \hat{e}_1 + \hat{e}_2 \rangle; \\ L_1 &= \langle -\hat{e}_1 + \hat{e}_2 \rangle; \\ L_2 &= \langle \hat{e}_3 \rangle. \end{aligned}$$

In this case, we have a *basis of eigenvectors.*

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Let's take some time with this.

$$p_P(t) = \det \begin{pmatrix} t & 0 & -1 & 0 \\ 0 & t-1 & 0 & 0 \\ -1 & 0 & t & 0 \\ 0 & 0 & 0 & t-1 \end{pmatrix} = (t-1) \det \begin{pmatrix} t & 0 & -1 \\ 0 & t-1 & 0 \\ -1 & 0 & t \end{pmatrix}.$$

We can still do column operations.

$$\begin{pmatrix} t & 0 & -1 \\ 0 & t-1 & 0 \\ -1 & 0 & t \end{pmatrix} \rightsquigarrow \begin{pmatrix} t-t^{-1} & 0 & -1 \\ 0 & t-1 & 0 \\ 0 & 0 & t \end{pmatrix},$$

whence

$$\det \begin{pmatrix} t & 0 & -1 \\ 0 & t-1 & 0 \\ -1 & 0 & t \end{pmatrix} = (t^2-1)(t-1).$$

(You're actually doing the column operations in the field $\mathbf{R}(t)$, which is the fraction field of the polynomial ring $\mathbf{R}[t]$. OOOH FANCY!)

In any case, we've got eigenvalues 1 and −1, and the eigenspaces are

$$\begin{aligned} L_1 &= \langle \hat{e}_2, \hat{e}_4, \hat{e}_1 + \hat{e}_3 \rangle; \\ L_{-1} &= \langle -\hat{e}_1 + \hat{e}_3 \rangle. \end{aligned}$$

Again we have a *basis of eigenvectors.*

Definition. We will say that an $n \times n$ matrix A is *diagonalizable* (over $\mathbf{R}$) if there exists a basis of $\mathbf{R}^n$ consisting of *real* eigenvectors for A.

This terminology may seem odd right now, but soon we will get to the bottom of it!

The examples we've thought about have located two obstructions to diagonalizability over $\mathbf{R}$

(1) *non-real eigenvalues*: the matrix $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has characteristic polynomial $t^2 + 1$. This has no real roots, so D has no real eigenvalues, and no real eigenvectors.

(2) *repeated eigenvalues, sometimes*: the matrices $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $A' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ each have characteristic polynomial $(t-2)^2$, but the eigenspace of the first is 1-dimensional, whereas the eigenspace of the second is 2-dimensional.

The first issue is really not such a big deal if you like complex numbers. We'll learn that we can make sense of linear algebra over the set of complex numbers, $\mathbf{C}$, as well, and then you have no problem finding a basis of $\mathbf{C}^2$ consisting of eigenvectors of the matrix D.

We say that D is diagonalizable over $\mathbf{C}$, but not over $\mathbf{R}$.

The second issue is more subtle. To understand it better, we have to understand *similarity*.

More generally, suppose I have a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of $\mathbf{R}^n$, and suppose A is an $n \times n$ matrix, giving us a linear map $T_A : \mathbf{R}^n \longrightarrow \mathbf{R}^n$.

Maybe T_A is actually more interesting to us than A, and maybe $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a better basis for us than the standard basis. So we want to express the action of T_A entirely in terms of $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

When we look at our chosen basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$, we can write each vector $T_A(\vec{v}_j)$ in a unique fashion as a linear combination of the basis vectors:

$$T_A(\vec{v}_j) = \sum_{i=1}^{n} \beta_{ij}\vec{v}_i.$$

We could have put all those coefficients together into a new matrix

$$B = (\beta_{ij}).$$

We say that B *represents* T_A *with respect to the basis* $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

If we'd done that with the standard basis, we'd have the matrix A staring back at us. But with a different basis, B isn't A. So how do they relate??

So, let's make a nice invertible matrix out of our basis:

$$V := \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}.$$

We see that

$$AV = \begin{pmatrix} \sum_{i=1}^n \beta_{i1}\vec{v}_i & \cdots & \sum_{i=1}^n \beta_{in}\vec{v}_i \end{pmatrix}.$$

On the other hand,

$$VB = \begin{pmatrix} \sum_{i=1}^n \beta_{i1}\vec{v}_i & \cdots & \sum_{i=1}^n \beta_{in}\vec{v}_i \end{pmatrix}.$$

So $AV = VB$, whence $B = V^{-1}AV$.

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This is a tricky concept. I like to think about this diagram:

$$\begin{array}{ccc}
\mathbf{R}^n_{\hat{e}_i} & \xrightarrow{A} & \mathbf{R}^n_{\hat{e}_i} \\
{\scriptstyle V}\uparrow & & \downarrow{\scriptstyle V^{-1}} \\
\mathbf{R}^n_{\vec{v}_i} & \xrightarrow[B]{} & \mathbf{R}^n_{\vec{v}_i}
\end{array}$$

Definition. We say two $n \times n$ matrices A and B are *similar* if they represent the same linear transformation with respect to two different bases.

Equivalently, A and B are similar if B represents T_A with respect to some other basis.

Equivalently, A and B are similar if and only if there is some invertible matrix V such that

$$B = V^{-1} A V.$$

Let's do a quick example. Consider $A = \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix}$, and let's write the matrix B that represents T_A with respect to the basis $\left\{ \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.

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$$
\begin{aligned}
B &= \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
$$

So our A is actually similar to a diagonal matrix.

And what does that mean? The matrix B that represents A with respect to $\{\vec{v}_1, \vec{v}_2\}$ is $\mathrm{diag}(4, -1)$, so:

$$A\vec{v}_1 = 4\vec{v}_1 \quad \text{and} \quad A\vec{v}_2 = -\vec{v}_2.$$

In other words, $\vec{v}_1$ and $\vec{v}_2$ form a *basis of eigenvectors* for A. So A is *diagonalizable*.

And now we understand our terminology: ***an $n \times n$ matrix is diagonalizable if and only if it is similar to a diagonal matrix.***