18.06.24: Eigenvalues

Lecturer: Barwick

Monday, 11 April 2016

Last time, we discovered that for any $n \times n$ matrix A with entries $a_{i,j}$, all but finitely many of the matrices $tI - A$ (for $t \in \mathbf{R}$) are invertible. ("Good things happen almost all the time.")

Let's think about the values of t fro which $tI - A$ is non-invertible; i.e., that $\det(tI - A) = 0$.

Let's think of this as a function on the reals:

$$p(t) = \det(tI - A).$$

Using formula for the determinant last time:

$$p(t) = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{n} \alpha_{\sigma(i),i}(t) \right),$$

where

$$\alpha_{\sigma(i),i}(t) = a_{\sigma(i),i} \quad \text{if } \sigma(i) \neq i,$$

and

$$\alpha_{\sigma(i),i}(t) = t - a_{i,i} \quad \text{if } \sigma(i) = i.$$

$$p(t) = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{n} \alpha_{\sigma(i),i}(t) \right),$$

where

$$\alpha_{\sigma(i),i}(t) = a_{\sigma(i),i} \quad \text{if } \sigma(i) \neq i,$$

and

$$\alpha_{\sigma(i),i}(t) = t - a_{i,i} \quad \text{if } \sigma(i) = i.$$

This formula isn't much for computation, but it tells us something qualitative: this function $p(t)$ is in fact a polynomial of degree n, called the ***characteristic polynomial*** of A.

Question. For how many t is $tI - A$ singular? That is, for how many t is $p(t) = \det(tI - A) = 0$?

Question. For how many t is $tI - A$ singular? That is, for how many t is $p(t) = \det(tI - A) = 0$?

Answer. At most n. □

What's going on here? We started with a matrix A, and we became curious (for no good reason) about when the matrix $tI - A$ is invertible.

The answer turned out to be: it's always invertible, except when t is one of the (at most n) roots of the polynomial $p(t) = \det(tI - A)$ of degree n.

Let's do an example:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Now the characteristic polynomial is

$$p(t) = \det(tI - A) = \begin{pmatrix} t-2 & -1 \\ -1 & t-2 \end{pmatrix} = t^2 - 4t + 3 = (t-3)(t-1).$$

So $tI - A$ is invertible unless $t \in \{1, 3\}$.

This is all nice, but it doesn't mean anything, does it … ?

Question. What's the significance of the number

0. 7390851332 1516064165 5312087673 8734040134 1175890075 7464965680 6357732846 5488354759 4599376106 9317665318 49801246 … ???

If you ever played with a calculator as a kid, you may have typed a number in (in radian mode) and hit "cos" a large number of times. It would stabilize around this value. This is the unique *fixed point* for cosine, i.e., the unique solution of the equation

$$\cos x = x.$$

An $n \times n$ matrix A tends not to have many *fixed vectors* (i.e., vectors $\vec{v}$ such that $A\vec{v} = \vec{v}$), except for $\vec{0}$.

For example, if $A = 3I$, then no nonzero vector is a fixed vector!

More generally, A has a nonzero fixed vector if and only if $A-I$ is noninvertible. And one of the things we learned is that *good things happen almost all the time*; in this case, $A - I$ is almost always invertible!

Since fixed vectors are pretty rare, it's not so interesting to look for them. But, in a sense, we can ask for *fixed directions* rather than fixed vectors.

In other words, we can look for *lines* $L \subseteq \mathbf{R}^n$ such that for any $\vec{v} \in L$, one has $A\vec{v} \in L$. In other words, we can ask about lines that are not moved by A.

Now a single nonzero vector $\vec{v} \in L$ spans L, of course, so when we say that $A\vec{v} \in L$, we're really saying that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbf{R}$, or, equivalently,

$$(\lambda I - A)\vec{v} = \lambda\vec{v} - A\vec{v} = \vec{0}.$$

But the only time that could ever happen is if $\lambda I - A$ has a nonzero kernel – or equivalently if $\lambda I - A$ is singular.

But wait. Didn't we just find out that there are only finitely many numbers $\lambda \in \mathbf{R}$ for which $\lambda I - A$ is singular? They're the roots of the characteristic polynomial

$$p(t) = \det(tI - A).$$

Let's look again at our matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We found out that $tI - A$ is invertible unless $t \in \{1, 3\}$. It's easy to see that

$$\dim\ker(I - A) = 1 \quad \text{and} \quad \dim\ker(3I - A) = 1.$$

So there are two lines L_1 and L_3 out there:

* every $\vec{v} \in L_1$ is fixed by A, so that $A\vec{v} = \vec{v}$;
* every $\vec{v} \in L_3$ is scaled by 3 by A, so that $A\vec{v} = 3\vec{v}$.

WIKIPEDIA ANIMATION.

Definition. If A is an $n \times n$ matrix, then any number $\lambda \in \mathbf{R}$ such that $tI - A$ is invertible – that is any λ that is a root of the polynomial $p(t) = \det(tI - A)$ – is called an ***eigenvalue*** of A.

If λ is an eigenvalue of A, then the subspace $\ker(\lambda I - A) \subseteq \mathbf{R}^n$ is called the ***eigenspace*** for A corresponding to λ.

If $\vec{v} \in \ker(\lambda I - A)$ is nonzero, then $\vec{v}$ is called an ***eigenvector*** with eigenvalue λ.

Question. What are the eigenvalues and eigenvectors of a diagonal matrix?