18.06.17: Fun with spacetime

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Last time, we started to prove that if $V \subseteq \mathbf{R}^n$ is a vector subspace, then $V^{\perp\perp} = V$. Let's finish up that.

Let's say that a covector $\underset{\rightarrow}{w} \in (\mathbf{R}^n)^\vee$ *annihilates* a vector $\vec{v} \in \mathbf{R}^n$ if and only if $\underset{\rightarrow}{w}\vec{v} = 0$. We'll also say that $\underset{\rightarrow}{w}$ *annihilates* a subspace $V \subseteq \mathbf{R}^n$ if and only if it annihilates every vector of V.

So $V^\perp \subseteq (\mathbf{R}^n)^\vee$ is the space of covectors that annihilate V, and $V^{\perp\perp}$ is the space of vectors that are annihilated by the covectors that annihilate V.

So one thing is clear: any vector of V lies in $V^{\perp\perp}$. So $V \subseteq V^{\perp\perp}$. But what about the other direction?

To show that the two subspaces are equal, it suffices to show that they have the same dimension. (Why? And how cool is that??)

To prove this, we'll need to know a key fact:

$$\dim(V^\perp) = n - \dim(V).$$

This is something that we'll need to spend some time to understand, and we're going to have to talk about notions of *orthogonality* and *orthonormality* to get the story straight. We'll get to all that after spring break, but first …

Here's the abstract statement of the Rank-Nullity Theorem:

(1) $\ker(A) = \text{coim}(A)^\perp$, so that

$$\dim(\ker(A)) = n - \dim(\text{coim}(A)).$$

(2) $\text{im}(A) = \text{coker}(A)^\perp$, so that

$$\dim(\text{im}(A)) = m - \dim(\text{coker}(A)).$$

(3) A provides a bijection $\text{coim}(A) \cong \text{im}(A)$, so that

$$\dim(\text{coim}(A)) = \dim(\text{im}(A)).$$

For the rest of today and tomorrow, I want to talk about *spacetime*.

Let's operate in the following model for our universe: take $\mathbf{R}^4$ (with standard basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4)$). We have the very important matrix

$$H = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we write $\eta(\vec{v}, \vec{w}) := \underset{\rightarrow}{v} H \vec{w}$.

(We choose units so that $c = 1$; so all speeds are measured as a fraction of the speed of light.)

For any vector $\vec{v} \in \mathbf{R}^4$, write $s^2(\vec{v}) = \eta(\vec{v}, \vec{v}) \in \mathbf{R}$. If $s^2(\vec{v}) > 0$, we say that $\vec{v}$ is *spacelike*; if $s^2(\vec{v}) < 0$, we say that $\vec{v}$ is *timelike*; if $s^2(\vec{v}) = 0$, we say that $\vec{v}$ is *lightlike*.

Let's draw a picture (forgetting the last coordinate if necessary) of the spacelike, timelike, and lightlike vectors. Explain this picture from a physical perspective.

A 4×4 matrix M such that

$$H = M^T H M.$$

is called a *Lorentz transformation*.

All Lorentz transformations are invertible. (Why?)

This means that a Lorentz transformation corresponds to a unique basis of $\mathbf{R}^4$; we may call such a basis a *Lorentz* basis. We may say that these bases are precisely those in which one may perform measurements and use the laws of physics. In other words, any physical laws we invent should be *invariant* under Lorentz transformations.

Any 3×3 *orthogonal* matrix R (i.e., a matrix R such that $R^T R = I$), the block matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix},$$

is a Lorentz transformation.

So if we choose a basis $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ for $\mathbf{R}^3$ such that $x_i \cdot x_j = \delta_{ij}$, then we get a Lorentz basis $(\hat{e}_1, \vec{x}_1, \vec{x}_2, \vec{x}_3)$. What is the physical interpretation of this?

There is a different sort of Lorentz basis as well. Consider an observer at the origin, moving in the positive x direction with speed u. This observer will agree with a stationary observer at the origin about the direction of the x, y, and z axes, and it will agree with the stationary observer's measurement of length in the y and z directions; however, this observer will see the x and t directions very differently…

Write $\phi := \tanh^{-1}(u)$. Prove that the matrix

$$\Lambda_\phi = \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) & 0 & 0 \\ -\sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a Lorentz transformation. This is the *Lorentz boost* in the positive x-direction at speed u. Physically, this means that an observer moving in the positive x direction with speed u will see a vector v in spacetime as $\Lambda_\phi v$.

This accounts for phenomena such as *time dilation* and *Lorentz contraction*. (How?)