18.06.15: ‘Rank-nullity: the sequel’

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Let's take a moment to imagine how our proof of the Rank-Nullity Theorem might have been different if we'd used column operations to get our matrix into rcef:

$$\textit{column operations}: A \rightsquigarrow AN,$$

where N is an invertible $n \times n$ matrix.

The point here is that *column operations don't change the image:*

$$\mathrm{im}(A) = \mathrm{im}(AN).$$

However, column operations absolutely *do* change the kernel:

$$\ker(A) \neq \ker(AN).$$

BUT, *column operations don't change the* ***dimension*** *of the kernel*:

$$\dim(\ker(A)) = \dim(\ker(AN)).$$

Using pure thought, tell me what the rank and nullity are of these matrices:

$$\begin{pmatrix} 5 & -15 \\ -2 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -138 \\ 5 & 1 & 75 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 6 & 3 \\ 5 & 1 & 50 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 9 & 9 \\ 1 & 1 & 1 \\ 4 & 4 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & 7 \\ -2 & 6 & 3 \\ -1 & 11 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix}$$

One final topic that we didn’t yet discuss. We’ve focused on solving equations $A\vec{x} = \vec{0}$, but what about the more general equation $A\vec{x} = \vec{v}$? What do we do there?

There are two options:

- *no solutions* – here $\vec{v}$ does not lie in the image of A;
- *at least one solution* – here $\vec{v} \in \operatorname{im}(A)$.

In the latter case, let’s try to work out a way to find *all* the solutions to $A\vec{x} = \vec{v}$.

Let’s suppose we’ve located one solution – a vector $\vec{x}_0 \in \mathbf{R}^n$ such that $A\vec{x}_0 = \vec{v}$. It turns out we can get all of them from that one, if we know about ... the kernel!

Why? Well, suppose $\vec{y} \in \ker(A)$. Then

$$A(\vec{x}_0 + \vec{y}) = A\vec{x}_0 + A\vec{y} = \vec{v} + \vec{0} = \vec{v}.$$

On the other hand, if $A\vec{x} = \vec{v}$, then

$$A(\vec{x} - \vec{x}_0) = A\vec{x} - A\vec{x}_0 = \vec{v} - \vec{v} = \vec{0}.$$

Hence the set of solutions to the equation $A\vec{x} = \vec{v}$ is the set

$$\{\vec{x} = \vec{x}_0 + \vec{y} \mid \vec{y} \in \ker(A)\}.$$

Let's do this in an example. If we have

$$\vec{v} = \begin{pmatrix} -4 \\ 3 \\ 7 \end{pmatrix}$$

and

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix},$$

let's find the set of solutions $\vec{x}$ to the equation $A\vec{x} = \vec{v}$.

The first step is to find a basis of $\ker(A)$:

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

On the other hand, it’s easy to find one solution:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

So, any solution can be written in a unique fashion as

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$