18.06.10: ‘Spaces of vectors’

Lecturer: Barwick

Friday 26 February 2016

Another day, another system of linear equations.

$$\begin{aligned} 0 &= -12x + 11y - 17z; \\ 0 &= 2x - y + 9z; \\ 0 &= -3x + 4y + 5z. \end{aligned}$$

Solve it!!

The rows are not linearly independent, so there are infinitely many solutions.

You can reduce these equations to just two: $41y = 37x$ and $41z = -5x$. In other words, any solution is a multiple of the vector

$$\begin{pmatrix} 41 \\ 37 \\ -5 \end{pmatrix}.$$

This is the information that the system of linear equations provides.

How infinite is infinite?

We’ve said a few times that a system of linear equations has either 0, 1, or $+\infty$ many solutions.

When the system of linear equations is of the form

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} a_{1i} x_i; \\
0 &= \sum_{i=1}^{n} a_{2i} x_i; \\
&\vdots \\
0 &= \sum_{i=1}^{n} a_{ni} x_i,
\end{aligned}
$$

we always have the solution $x_1 = x_2 = \cdots = x_n = 0$, so our only two options in this case are 1 or $+\infty$. But we can we say more. For example, are all the solutions multiples of a single vector??

Here's a system of linear equations with infinitely many solutions:

$$\begin{aligned} 0 &= 3x - 2y; \\ 0 &= 4y - 5z; \\ 0 &= 6x - 5z. \end{aligned}$$

We reduce to $3x = 2y$ and $4y = 5z$, and that's all the information. The last equation doesn't actually participate. So any vector that satisfies the system above is a multiple of

$$\begin{pmatrix} 1 \\ 2/3 \\ 8/15 \end{pmatrix}.$$

Here’s a system of linear equations with infinitely many solutions:

$$\begin{aligned} 0 &= 4u + 2v + 6x + 3y; \\ 0 &= 2u + 3x \\ 0 &= 2v + 3y; \\ 0 &= 2u - 4v + 3x - 6y. \end{aligned}$$

How close can you come?

You can see straightaway that the equations $2u = -3x$ and $2v = -3y$ completely determine the system. The other equations are just offering the same information. So any vector that satisfies the system above is a linear combination of

$$\begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 3 \\ 0 \\ -2 \end{pmatrix}.$$

This lets us parametrize the solution space!!

What we’re doing when we solve systems of linear equations is finding a basis for the space of solutions.

Definition. A *subspace* $V \subseteq \mathbf{R}^n$ is a collection V of vectors of $\mathbf{R}^n$ such that:

(1) for any vectors $\vec{v}, \vec{w} \in V$, the sum $\vec{v} + \vec{w} \in V$;

(2) for any real number r and any vectors $\vec{v} \in V$, the scalar multiple $r\vec{v} \in V$.

Example. For any $m \times n$ matrix A, the set

$$\ker(A) := \{\vec{v} \in \mathbf{R}^n \mid A\vec{v} = \vec{0}\}$$

is a subspace of $\mathbf{R}^n$. This is called the *kernel* of A. This may also be called the *space of solutions* of the system of linear equations:

$$\begin{aligned} 0 &= \sum_{i=1}^{n} a_{1i} x_i; \\ &\vdots \\ 0 &= \sum_{i=1}^{n} a_{ni} x_i. \end{aligned}$$

Definition. A *basis* of a subspace $V \subseteq \mathbf{R}^n$ is a collection $\{\vec{v}_1, \ldots, \vec{v}_k\}$ of vectors $\vec{v}_i \in V$ such that:

(1) the vectors $\vec{v}_1, \ldots, \vec{v}_k$ are *linearly independent*;

(2) the vectors $\vec{v}_1, \ldots, \vec{v}_k$ *span* V.

We say that V is *k-dimensional*.

The two conditions in the definition above are complementary. To illustrate, let's write them this way.

(1) The vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ are *linearly independent* if and only if, for any vector $\vec{w} \in V$, there exists *at most one* way to write $\vec{w}$ as a linear combination

$$\vec{w} = \sum_{i=1}^{k} \alpha_i \vec{v}_i.$$

(2) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ *span* V if and only if, for any vector $\vec{w} \in V$, there exists *at least one* way to write $\vec{w}$ as a linear combination

$$\vec{w} = \sum_{i=1}^{k} \alpha_i \vec{v}_i.$$

(3) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are a *basis* of V if and only if, for any vector $\vec{w} \in V$, there exists *exactly one* way to write $\vec{w}$ as a linear combination

$$\vec{w} = \sum_{i=1}^{k} \alpha_i \vec{v}_i.$$

The similarities between these conditions and the conditions of injectivity, surjectivity, and bijectivity are no accident…

Take some vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ and make them into the columns of an $n \times k$ matrix

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{pmatrix}.$$

Multiplication by A is a map $T_A : \mathbf{R}^k \longrightarrow V$ that carries $\hat{e}_i$ to $\vec{v}_i$.

(1) The vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ are *linearly independent* if and only if T_A is injective.

(2) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ *span* V if and only if T_A is surjective.

(3) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are a *basis* of V if and only if T_A is bijective.