18.06.03: ‘Length and angle’

Lecturer: Barwick

Monday 08 February 2016

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The *length* of a vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbf{R}^n$ is defined using the good old law of Pythagoras:

$$\|\vec{a}\| \coloneqq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Geometrically, it works exactly as we expect – it's the length of the line segment from the origin to $(a_1, a_2, \ldots, a_n)$:

Here’s a vector in $\mathbf{R}^9$. How long is it?

$$\vec{v} = \begin{pmatrix} -6 \\ 0 \\ 2 \\ -3 \\ 4 \\ -7 \\ 0 \\ 9 \\ -1 \end{pmatrix}$$

$$\left\| \begin{pmatrix} -6 \\ 0 \\ 2 \\ -3 \\ 4 \\ -7 \\ 0 \\ 9 \\ -1 \end{pmatrix} \right\| = 14$$

Here are the key facts about length, which you can deduce either from the geometry or from some easy algebra:

- $\|\vec{a}\| \geq 0$;
- $\|\vec{a}\| = 0$ if and only if $\vec{a} = \vec{0}$;
- $\|r\vec{a}\| = |r|\|\vec{a}\|$;
- $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.

The story we're told is that *nonzero vectors are nothing more than a length and a direction.* We've already sorted out what length is. Direction can be obtained, in effect, by normalizing the length: if $\vec{a} \neq \vec{0}$, then we define

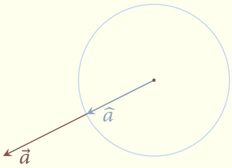
$$\widehat{a} \coloneqq \frac{1}{\|\vec{a}\|}\vec{a}.$$

This is the *unit vector in the direction of* $\vec{a}$. Clearly $\|\widehat{a}\| = 1$, and so

$$\vec{a} = \|\vec{a}\|\widehat{a}.$$

The unit vector $\widehat{a}$ is the direction of $\vec{a}$.

One way to think about this is that you take a nonzero vector, and scale it so that the head of the arrow lies at the unit circle:



Remember that vector $\vec{v} \in \mathbf{R}^9$? The corresponding unit vector is

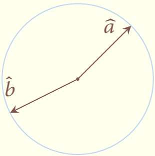
$$\widehat{v} = \begin{pmatrix} -3/7 \\ 0 \\ 1/7 \\ -3/14 \\ 2/7 \\ -1/2 \\ 0 \\ 9/14 \\ -1/14 \end{pmatrix}.$$

There are unit vectors in $\mathbf{R}^n$ that everyone knows; we mentioned them on Friday:

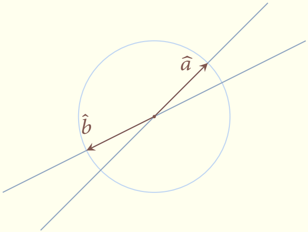
$$\hat{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the 1 is in the i-th spot.

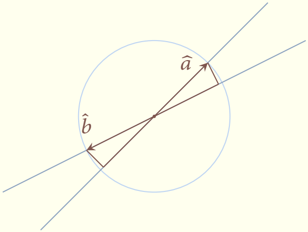
While walking in a quiet area of $\mathbf{R}^{17}$, you encounter two unit vectors. Since these two vectors define a plane, you can just look at that plane:



You can also draw the lines in the plane these vectors define:



Now we can drop perpindiculars from our unit vectors to these lines:

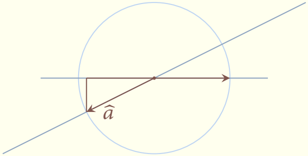


The *signed* length from the origin to the right angle (either one!!) is the *dot product* $\hat{a} \cdot \hat{b}$. Note that this is a *scalar* (not a vector), and it is the cosine of the angle between $\hat{a}$ and $\hat{b}$.

Let’s get some easy stuff out of the way. If $\hat{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, then what is

$$\hat{a} \cdot \hat{e}_i = ?$$

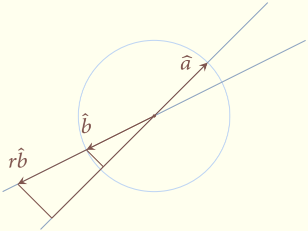
Indeed, $\widehat{a} \cdot \hat{e}_i = a_i$:



But what if $\vec{a}$ and $\vec{b}$ aren't unit vectors?

In that case, we scale our dot product accordingly:

$$\vec{a} \cdot \vec{b} := \|\vec{a}\|\|\vec{b}\|(\hat{a} \cdot \hat{b}).$$



This also gives us another way of writing down the length:

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}.$$

More generally, you have a good distributive law:

$$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}.$$

This distributive law is actually the key to computing the dot product!

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (a_1 \hat{e}_1 + a_2 \hat{e}_2 + \cdots + a_n \hat{e}_n) \cdot \vec{b} \\ &= a_1(\hat{e}_1 \cdot \vec{b}) + a_2(\hat{e}_2 \cdot \vec{b}) + \cdots + a_n(\hat{e}_n \cdot \vec{b}) \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \end{aligned}$$

It’s surprisingly good news that the formula is so simple! Let’s see if we can use this.

Question. What’s the angle between the following two lines in $\mathbf{R}^4$?

$$l_1(t) = (t, -t, t, -t) \quad \text{and} \quad l_2(t) = (2t, t, 0 - 2t)$$

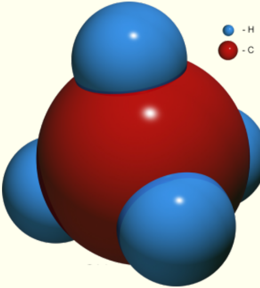
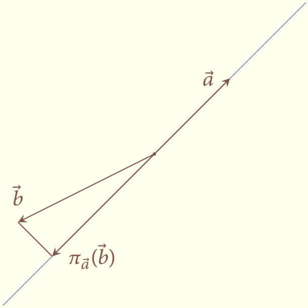


Figure 1: Methane

Question. What are the angles between the bonds in a molecule of methane?

The *projection* of a vector $\vec{b}$ onto a vector $\vec{a}$ is the vector

$$\pi_{\vec{a}}(\vec{b}) \coloneqq (\hat{a} \cdot \vec{b})\hat{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}.$$



On Wednesday, we will use the dot product repeatedly to convert systems of linear equations into matrices.

The first problem set will be online soon.