# 18.06 Final Exam

19 May 2016 at 9 AM

STATE YOUR NAME: Donald J. Trump

CIRCLE YOUR RECITATION: R666 0–1 SATAN

GRADING

| | | |
|---|---|---|
| 1. | 20 | /20 |
| 2. | 20 | /20 |
| 3. | 20 | /20 |
| 4. | 20 | /20 |
| 5. | 20 | /20 |
| 6. | 20 | /20 |
| 7. | 20 | /20 |
| 8. | 20 | /20 |
| TOTAL | 160 | /160 |

## 1. Clinton or Trump

For each of the following sentences, indicate whether they are true or false. (No need to justify your answer.)

(a) If $A$ is an $n \times n$ matrix with characteristic polynomial $p_A(t) = t^n$, then $A = 0$.

FALSE

(b) If $A$ is a matrix, then any element of the kernel of $A$ is perpindicular to any element of the image of $A^\intercal$.

TRUE

(c) The only $m \times n$ matrix of rank 0 is 0.

TRUE

(d) There is a orthogonal basis of $\mathbf{C}^3$ consisting of eigenvectors for the matrix

$$\begin{pmatrix} 17822 & -759i & -14795 + 69532i \\ 759i & 568347 & 385955 \\ -14795 - 69532i & 385955 & 10479 \end{pmatrix}.$$

TRUE

(e) If

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an $2n \times 2n$ matrix in which $A$, $B$, $C$, and $D$ are all $n \times n$ blocks, then

$$\det M = (\det A)(\det D) - (\det B)(\det C).$$

FALSE

## 2. Solve

Write a basis for the space of solutions to the system of linear equations

$$\begin{aligned} a + b + 2c + 4d + 7e &= 0; \\ a + 2b + 4c + 7d + 13e &= 0; \\ 2a + 4b + 7c + 13d + 24e &= 0; \\ 4a + 7b + 13c + 24d + 44e &= 0. \end{aligned}$$

*Solution.* We seek a basis for the kernel of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 & 7 \\ 1 & 2 & 4 & 7 & 13 \\ 2 & 4 & 7 & 13 & 24 \\ 4 & 7 & 13 & 24 & 44 \end{pmatrix}.$$

I note that the first three columns $A^1, A^2, A^3$ are linearly independent, $A^4 = A^1 + A^2 + A^3$, and $A^5 = A^2 + A^3 + A^4$. So the rank is 3, and the nullity is two. Now I can do some easy column operations:

$$\left(\begin{array}{ccccc} 1 & 1 & 2 & 4 & 7 \\ 1 & 2 & 4 & 7 & 13 \\ 2 & 4 & 7 & 13 & 24 \\ 4 & 7 & 13 & 24 & 44 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc} 1 & 1 & 2 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 \\ 2 & 4 & 7 & 0 & 0 \\ 4 & 7 & 13 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

and there's my basis:

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

□

## 3. Project

Compute the projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbf{R}^4$ onto the kernel of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

*Solution.* The kernel is the same as the kernel of

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

which is the orthogonal complement to the image of $A^\intercal$. So if $v = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, then the projection we want is

$$\pi_{\ker(A)}(v) = v - \pi_{\mathrm{im}(A^\intercal)}(v) = v - A^\intercal(AA^\intercal)^{-1}Av,$$

whence:

$$\pi_{\ker(A)}(v) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ -1 \end{pmatrix},$$

whence

$$\pi_{\ker(A)}(v) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3/2 \\ 2 \\ 3/2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1 \\ 1/2 \\ 1 \end{pmatrix}.$$

$\square$

## 4. Charlie Brown

Compute the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 \end{pmatrix}.$$

*Solution.* This matrix $X$ is a block matrix, and the blocks are:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & -9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}.$$

And their inverses are easy to compute:

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -19 & -5 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -3 & 15 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},$$

so that gives

$$X^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -19 & -5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \end{pmatrix}. \qquad \square$$

## 5. The Outer Limits

Everyone's favorite matrix is built like this: take a unit vector $\hat{x} \in \mathbf{R}^n$, and set $P \coloneqq \hat{x}\hat{x}^\intercal$. In terms of $\hat{x}$, describe the kernel of $P$.

*Solution.* $P$ is the orthogonal projection onto the line spanned by $\hat{x}$. Hence $\ker(P) = \hat{x}^\perp$, the orthogonal complement of the line spanned by $\hat{x}$. ☐

What are the nonzero eigenvalues of $P$?

*Solution.* There is only one: 1. Of course 0 is an eigenvalue of $P$ with eigenspace $\hat{x}^\perp$, and since $P$ is symmetric, it has an orthogonal basis of eigenvectors (Spectral Theorem!), so $\hat{x}$ spans the eigenspace of the only other eigenvalue, 1. ☐

What are the corresponding eigenspaces?

*Solution.* The line spanned by $\hat{x}$. ☐

## 6. Corny crony

Compute the characteristic polynomial of

$$\begin{pmatrix} 0 & 0 & 8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}.$$

*Solution.* We have blocks given by companion marices. So the characteristic polynomial is $(t^3 - 8)(t^3 + 3t^2 + 3t + 1)$. □

## 7. Permute

Is the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

diagonalizable over **R**? over **C**?

*Solution.* We have a block matrix. The first block,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

has characteristic polynomial $t^4 - 1$. The second block,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

has characteristic polynomial $t^2 - 1$. Since each block has distinct eigenvalues, and since not all of them are real in the first block, the matrix is *not* diagonalizable over **R**, but *is* diagonalizable over **C**. □

## 8. You'll flip

Contemplate the following matrix

$$A = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix}.$$

Before you compute anything, is this matrix diagonalizable over **R**? over **C**? How do you know?

*Solution.* It is diagonalizable over each. The eigenvalues are real, and there's an orthogonal basis of eigenvectors by the Spectral Theorem. □

Now compute the eigenvalues and eigenspaces of this matrix.

*Solution.* The characteristic polynomial is

$$\det(tI-A) = \det \begin{pmatrix} t-5 & 1 & 1 \\ 1 & t-5 & 1 \\ 1 & 1 & t-5 \end{pmatrix} = t^3-15t^2+72t-108 = (t-3)(t-6)^2.$$

So we have eigenvalues 3 and 6. The eigenspace for 3 is

$$\ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

is 1-dimensional, so it's spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The eigenspace for 6 is the orthogonal complement of that vector, which is the image of the transpose of the matrix above, which is 2-dimensional space spanned by $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. □