# 18.06 Exam III: Orthogonalize this!

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| GRADING | |
|---|---|
| 1. 20 | /20 |
| 2. 20 | /20 |
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| 5. 20 | /20 |
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## 1. Veracious or fallacious

For each of the following sentences, indicate whether they are true or false. (No need to justify your answer.)

(a) If $\vec{v} \in \mathbf{R}^n$ is a vector and $W \subseteq \mathbf{R}^n$ is a vector subspace, then the projection $\pi_W(\vec{v}) = \vec{0}$ if and only if, for any vector $\vec{w} \in W$, one has $\vec{v} \cdot \vec{w} = 0$.
TRUE.

(b) If $\vec{v} \in \mathbf{R}^n$ is a vector and $W \subseteq \mathbf{R}^n$ is a vector subspace, then

$$\|\pi_W(\vec{v})\| \leq \|\vec{v}\|.$$

TRUE.

(c) Two vector subspaces $V, W \subset \mathbf{R}^n$ such that $V \cap W = \{\vec{0}\}$ are othrogonal.
FALSE.

(d) Any vector subspace $W \subseteq \mathbf{R}^n$ has an orthonormal basis.
TRUE.

(e) The only orthonormal basis of $\mathbf{R}^n$ is the standard basis $\hat{e}_1, \ldots, \hat{e}_n$.
FALSE.

## 2. Solve

Find an orthogonal basis for the space of solutions to the following system of linear equations in the five variables $u, v, w, x, y$:

$$\begin{aligned} u + w + y &= 0 \\ v + x &= 0 \end{aligned}$$

*Solution.* Let's use column operations to compute a basis of the kernel of $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$:

$$\left(\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

So we have a basis

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

which we have to orthogonalize. But $\vec{v}_1$ and $\vec{v}_2$ are already orthogonal, so we set $\vec{w}_1 = \vec{v}_1$ and $\vec{w}_2 = \vec{v}_2$, and we only have to worry about fixing $\vec{v}_3$. Now $\vec{v}_3$ is already orthogonal to $\vec{v}_2$, so we have

$$\begin{aligned} \vec{w}_3 &= \vec{v}_3 - \pi_{\vec{w}_1}(\vec{v}_3) \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

And $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is our desired orthogonal basis. □

## 3. Is this projection accurate?

What is the projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbf{R}^3$ onto the plane $3x - 4y + z = 0$?

*Solution.* The vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies on the plane $3x - 4y + z = 0$, so its projection onto that plane is simply $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ again. □

## 4. More projecting

Compute the projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbf{R}^5$ onto the image of the following matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

*Solution.* Let's call the vector $\vec{b}$ and the matrix $A$. The columns of $A$ are linearly independent, so we'll compute the projection using the formula

$$\pi_{\operatorname{im} A}(\vec{b}) = A(A^\intercal A)^{-1} A^\intercal \vec{b}.$$

We have

$$A^\intercal \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

We get

$$A^\intercal A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

which is extremely easy to invert: we get

$$(A^\intercal A)^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix},$$

and so

$$(A^\intercal A)^{-1} A^\intercal \vec{b} = \begin{pmatrix} 1/2 \\ -1/4 \\ 1 \end{pmatrix}.$$

Now we get

$$\pi_{\operatorname{im} A}(\vec{b}) = A(A^\intercal A)^{-1} A^\intercal \vec{b} = \begin{pmatrix} 1/3 \\ 1 \\ -1/3 \\ 1 \\ 1/3 \end{pmatrix}.$$

□

## 5. Householder

Suppose $\hat{x} \in \mathbf{R}^n$ a unit vector. Write

$$N = \{\vec{v} \in \mathbf{R}^n \mid \vec{v} \cdot \hat{x} = 0\} \subset \mathbf{R}^n.$$

This $N$ is an $(n-1)$-dimensional vector subspace of $\mathbf{R}^n$. Also, write $H$ for the $n \times n$ matrix $I - 2\hat{x}\hat{x}^\intercal$.

Prove that the projection $\pi_N(\vec{w})$ of $\vec{w}$ onto $N$ is equal to the projection $\pi_N(H\vec{w})$ of $H\vec{w}$ onto $N$.

*Solution.* Suppose $\vec{v}_1, \dots, \vec{v}_{n-1}$ a basis of $N$; each of these vectors is perpindicular to $\hat{x}$, so that

$$\vec{v}_i^\intercal \hat{x} = \vec{v}_i \cdot \hat{x} = 0.$$

Now if $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_{n-1} \end{pmatrix}$, then we have

$$\begin{aligned} \pi_N(H\vec{w}) &= A(A^\intercal A)^{-1} A^\intercal (I - 2\hat{x}\hat{x}^\intercal)\vec{w} \\ &= A(A^\intercal A)^{-1} A^\intercal \vec{w} - 2A(A^\intercal A)^{-1} A^\intercal \hat{x}\hat{x}^\intercal \vec{w}. \end{aligned}$$

Since

$$\pi_N(\vec{w}) = A(A^\intercal A)^{-1} A^\intercal \vec{w},$$

we want to show that

$$2A(A^\intercal A)^{-1} A^\intercal \hat{x}\hat{x}^\intercal \vec{w} = \vec{0}.$$

But this is true: we have

$$A^\intercal \hat{x} = \begin{pmatrix} \vec{v}_1^\intercal \hat{x} \\ \vdots \\ \vec{v}_{n-1}^\intercal \hat{x} \end{pmatrix} = \vec{0},$$

and the proof is complete. □