18.06 Professor Strang Quiz 3 – Solutions May 7th, 2012

**Grading**

**Your PRINTED name is:** ____________________

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**Please circle your recitation:**

| | | | | |
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| r04 | T 12 | 36-153 | Rune Haugseng | haugseng |
| r05 | T 1 | 4-153 | Dimiter Ostrev | ostrev |
| r06 | T 1 | 4-159 | Uhi Rinn Suh | ursuh |
| r07 | T 1 | 66-144 | Ailsa Keating | ailsa |
| r08 | T 2 | 66-144 | Niels Martin Moller | moller |
| r09 | T 2 | 4-153 | Dimiter Ostrev | ostrev |
| r10 | ESG | | Gabrielle Stoy | gstoy |

**1 (33 pts.)**

Suppose an $n \times n$ matrix $A$ has $n$ independent eigenvectors $x_1, \ldots, x_n$. Then you could write the solution to $\frac{du}{dt} = Au$ in three ways:

$$u(t) = e^{At}u(0), \quad \text{or}$$

$$u(t) = Se^{\Lambda t}S^{-1}u(0), \quad \text{or}$$

$$u(t) = c_1 e^{\lambda_1 t}x_1 + \ldots + c_n e^{\lambda_n t}x_n.$$

Here, $S = [x_1 \mid x_2 \mid \ \ldots \ \mid x_n]$.

(a) From the definition of the exponential of a matrix, show why $e^{At}$ is the same as $Se^{\Lambda t}S^{-1}$.

*Solution.* Recall that $A = S\Lambda S^{-1}$, and $A^k t^k = S\Lambda^k t^k S^{-1}$. Then, definition of the exponential:

$$\boxed{\exp(At) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = S\left(\sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!}\right)S^{-1} = Se^{\Lambda t}S^{-1}.}$$

□

(b) How do you find $c_1, \ldots, c_n$ from $u(0)$ and $S$?

*Solution.* Since $e^0 = 1$, we see that

$$u(0) = c_1 x_1 + \ldots + c_n x_n = S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

where we used the definition of the matrix product. Thus the answer is:

$$\boxed{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}u(0).}$$

□

(c) For this specific equation, write $u(t)$ in any one of the (added: latter two of the) three forms, using *numbers* not symbols: You can choose which form.

$$\frac{du}{dt} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} u, \quad \text{starting from} \quad u(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

*Solution.* We diagonalize $A$ and get:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Thus $c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so for the second form

$$\boxed{u(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix},}$$

while in the third form:

$$\boxed{u(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.}$$

□

**2 (30 pts.)**

This question is about the real matrix

$$A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}, \quad \text{for} \quad c \in \mathbb{R}.$$

(a) - Find the eigenvalues of $A$, depending on $c$.

- For which values of $c$ does $A$ have real eigenvalues?

*Solution.* Since $0 = \mathrm{tr} A = \lambda_1 + \lambda_2$, we see that $\lambda_2 = -\lambda_1$.

Also, $-1 - c = \det A = -\lambda_1^2$. Thus,

$$\boxed{\lambda = \pm\sqrt{1+c}.}$$

Therefore,

$$\boxed{\text{the eigenvalues are real precisely when } c \geq -1.}$$

□

(b) - For one particular value of $c$, convince me that $A$ is similar to both the matrix

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

and to the matrix

$$C = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- Don't forget to say which value $c$ this happens for.

*Solution.* If two matrices are similar, then they do have the same eigenvalues (those are $2, -2$ for both $B$ and $C$). Here we must therefore have $0 = \text{tr}A$ and $-1 - c = \det A = -4$. We see that this happens precisely when $c = 3$, where we check that indeed the eigenvalues are $2, -2$. However, this does not guarantee that they are similar - and hence is not convincing.

Convincing: The eigenvalues $2, -2$ are different, so both $A$, $B$ and $C$ are diagonalizable, with the same diagonal matrix (for example to $\Lambda = B$!). Therefore $A$, $B$ and $C$ are all similar when $c = 3$. □

(c) For one particular value of $c$, convince me that $A$ cannot be diagonalized. It is not similar to a diagonal matrix $\Lambda$, when $c$ has that value.

- Which value c?

- Why not?

*Solution.* As we saw above, $\text{tr}A = 0$, so regardless of $c$ the eigenvalues come in pairs $\lambda_2 = -\lambda_1$. This means that whenever $\lambda_1 \neq 0$, we have two different eigenvalues, and hence $A$ is diagonalizable (not what we're after).

Thus we need $\lambda_1 = \lambda_2 = 0$, a repeated eigenvalue, which happens when $c = -1$ (so $\det A = 0$) as the only suspect – does it work?

$$\boxed{\text{Convincing: For } c = -1, \text{ we have } N(A - 0 \cdot I) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}$$

With only a 1-dimensional space of eigenvectors for the matrix, we are convinced that $A$ is not diagonalizable for $c = -1$. $\square$

**3 (37 pts.)**

(a) Suppose $A$ is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

- What is the largest number real number $c$ that can be subtracted from the diagonal entries of $A$, so that $A - cI$ is positive semidefinite?

- Why?

*Solution.* - We first realize that: If $A$ is symmetric, then $A - cI$ is also symmetric, since in general $(A + B)^T = A^T + B^T$ (simple, but very important to check!).

- Then we realize that the eigenvalues of $A - cI$ are $\lambda_1 - c \leq \lambda_2 - c \leq \ldots \leq \lambda_n - c$. Therefore:

$$\boxed{c = \lambda_1 \text{ is the largest that can ensure positive semidefiniteness (and it does).}}$$

□

(b) Suppose $B$ is a matrix with independent columns.

- What is the nullspace $N(B)$?
- Show that $A = B^T B$ is positive definite. Start by saying what that means about $x^T Ax$.

*Solution.* - Then $Bx = 0$ only has the zero solution, so $N(B) = \{0\}$.

- Again, we start by observing that $A^T = A$ is symmetric. Then we recall what positive definite means (the "energy" test):

$$\boxed{x^T Ax > 0 \quad \text{whenever} \quad x \neq 0.}$$

Thus, we see here (by definition the inner product property of the transpose of a matrix):

$$\boxed{x^T Ax = x^T(B^T(Bx)) = (Bx)^T(Bx) = \|Bx\|^2 \geq 0.}$$

So $A = B^T B$ is positive semidefinite. But finally, the equality $\|Bx\|^2 = 0$, only happens when $Bx = 0$ which by $N(B) = \{0\}$ means $x = 0$. □

(c) This matrix $A$ has rank $r = 1$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

- Find its largest singular value $\sigma$ from $A^T A$.

- From its column space and row space, respectively, find unit vectors $u$ and $v$ so that

$$Av = \sigma u, \quad \text{and} \quad A = u\sigma v^T.$$

- From the nullspaces of $A$ and $A^T$ put numbers into the full SVD (Singular Value Decomposition) of $A$:

$$A = \begin{bmatrix} | & & | \\ u & & \dots \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma & & 0 \\ & & \\ 0 & & \dots \end{bmatrix} \begin{bmatrix} | & & | \\ v & & \dots \\ | & & | \end{bmatrix}^T.$$

*Solution.* We compute:

$$A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}.$$

Thus the two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 10$, and $\sigma = \sqrt{10}$. For $v$, we find a vector in $N(A^T A - 10I)$, and normalize to unit length:

$$v = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Then we find $u$ using

$$u = \frac{Av}{\sigma} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

Since we have the orthogonal sums of subspaces $\mathbb{R}^2 = \mathbb{R}^m = c(A) \oplus N(A^T)$ and also $\mathbb{R}^2 = \mathbb{R}^n = c(A^T) \oplus N(A)$, we need to find one unit vector from each of $N(A)$ and $N(A^T)$ and augment to $v$ and $u$, respectively:

$$v_2 = \begin{bmatrix} 1\sqrt{2} \\ -1\sqrt{2} \end{bmatrix} \in N(A),$$

$$u_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \in N(A^T),$$

Thus, we finally see the full SVD:

$$\boxed{A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{5} & -2\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T.}$$

We remember, as a final check, to verify that the square matrices $U$ and $V$ both contain orthonormal bases of $\mathbb{R}^2$ as they should:

$$UU^T = I_2,$$

$$VV^T = I_2.$$

$\square$