18.06 Spring 2012 – Problem Set 7

This problem set is due Thursday, April 19th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 2 from Section 8.3.

   Solution. Since $\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are the eigenvector vectors for the eigenvalues 1 and 0.75, respectively,

   $$S = \begin{bmatrix} 0.6 & -1 \\ 0.4 & 1 \end{bmatrix}.$$

   A^k approches to

   $$S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}.$$

   □

2. Do Problem 7 from Section 8.3 (do also the "challenge problem" part).

   Solution. The eigenvalues are 1 and 0.5, and the eigenvectors are

   $$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

   Since

   $$A^k = S \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}^k S^{-1},$$

   for

   $$S = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix},$$

   $A^\infty = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}.$

   Challenge problem Let $A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$, $0 \le a, b \le 1$, be a Markov Matrix with steady state $\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$. Then

   $$A \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

Hence $0.6a + 0.4b = 0.6$. In other words,

$$A = \begin{bmatrix} 0.6 + 0.4x & 0.4 - 0.4x \\ 0.6 - 0.6x & 0.4 + 0.6x \end{bmatrix}$$

for some $-\frac{2}{3} \le x \le 1$.

□

3. Do Problem 9 from Section 8.3.

Solution. If every entry of A is nonnegative, every entry of A^2 is also nonnegative. Since, for any $j = 1, \cdots, n$, $\sum_i (A)_{ij} = 1$,

$$\sum_i (A^2)_{ij} = \sum_{i,k} a_{ik} a_{kj} = \sum_k \sum_i a_{ik} a_{kj} = \left(\sum_k \left(\sum_i a_{ik} \right) a_{kj} \right) = \sum_k a_{kj} = 1.$$

□

4. Do Problem 12 from Section 8.3.

Solution. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -0.5$. We have a steady state for the Markov matrix A. For the steady state v, $(A - I)v = 0 = 0v$. So $A - I$ have $\lambda = 0$. If $u_t = e^{\lambda_1 t} c_1 x_1 + e^{\lambda_2 t} c_2 x_2$ for the initial value $u_0 = c_1 x_1 + c_2 x_2$, u_t converges to $c_1 x_1$ as $t \to \infty$. □

5. Do Problem 4 from Section 6.3.

Solution. $v + w$ is constant if and only if $\frac{d(v+w)}{dt} = 0$.

$$\frac{d(v+w)}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = (w - v) + (v - w) = 0,$$

so $v + w$ is constant.

Let $u = \begin{bmatrix} v \\ w \end{bmatrix}$. Then

$$\frac{du}{dt} = \begin{bmatrix} \frac{dv}{dt} \\ \frac{dw}{dt} \end{bmatrix} = \begin{bmatrix} w - v \\ v - w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

The eigenvalue of $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ is given by solving $\det(A - \lambda I) = 0$. $\det(A - \lambda I) = (-1-\lambda)^2 - 1 = 1 + 2\lambda + \lambda^2 - 1 = \lambda(\lambda + 2)$ so the eigenvalues are $\lambda_1 = 0, \lambda_2 = -2$. We then observe that the corresponding eigenvectors are $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively.

Then the pure exponential solutions are given by

$$u_1(t) = e^{\lambda_1 t} x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u_2(t) = e^{\lambda_2 t} x_2 = e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the complete solutions are given by

$$u(t) = C \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} C + De^{-2t} \\ C - De^{-2t} \end{bmatrix}.$$

From the initial condition that $u(0) = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$, we get $C = 20$, $D = 10$.

That is,

$$v(t) = 20 + 10e^{-2t}$$
$$w(t) = 20 - 10e^{-2t}.$$

So $v(1) = 20 + 10e^{-2}, w(1) = 20 - 10e^{-2}, v(\infty) = w(\infty) = 20$.

□

6. Do Problem 5 from Section 6.3.

Solution. Now we have

$$\frac{du}{dt} = -Au = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u.$$

The eigenvalues of $-A$ are given by -1 times the eigenvalues of A, so now we have $\lambda_1 = 0, \lambda_2 = 2$. The corresponding eigenvectors are the same as those of A, namely $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then the pure exponential solutions are given by

$$u_1(t) = e^{\lambda_1 t} x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u_2(t) = e^{\lambda_2 t} x_2 = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the complete solutions are given by

$$u(t) = C \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} C + De^{2t} \\ C - De^{2t} \end{bmatrix}.$$

From the initial conditions $u(0) = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$, we get $C = 20$, $D = 10$, and $v(t) = 20 + 10e^{2t}$. So as $t \to \infty$, $v \to \infty$.

□

7. Do Problem 12 from Section 6.3.

Solution. Substituting $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ gives

$$\lambda^2 e^{\lambda t} = 6\lambda e^{\lambda t} - 9e^{\lambda t},$$

so $e^{\lambda t}(\lambda - 3)^2 = 0$, which means $\lambda = 3$ is a repeated root.

In terms of the matrix equation, since the matrix has trace 6 and determinant 9, its only eigenvalue is 3, with one independent eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To show that $y = te^{3t}$ is the second solution, just substitute this into the original differential equation. Since we have:

$$\begin{aligned} y' &= e^{3t} + 3te^{3t} \\ y'' &= 3e^{3t} + (3e^{3t} + 9te^{3t}) = 6e^{3t} + 9te^{3t}. \end{aligned}$$

Also,

$$6y' - 9y = 6e^{3t} + 18te^{3t} - 9te^{3t} = 6e^{3t} + 9te^{3t},$$

so we see that $y'' = 6y' - 9y$ when $y = te^{3t}$.

□

8. Do Problem 24 from Section 6.3.

Solution. A is an upper-triangular matrix, so we can read off its eigenvalues as the diagonal entries: 1, 3. By inspection we see that $(1, 0)$ is an eigenvector with eigenvalue 1. To find an eigenvector with eigenvalue 3 we observe

$$A - 3I = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix},$$

and so $(1, 2)$ is in its nullspace. Thus

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

and

$$A = S\Lambda S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$e^{At} = Se^{\Lambda t}S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & \frac{1}{2}(e^{3t} - e^t) \\ 0 & e^{3t} \end{pmatrix}.$$

When $t = 0$ this is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, as expected. Differentiating with respect to t, we get

$$\begin{pmatrix} e^t & \frac{1}{2}(3e^{3t} - e^t) \\ 0 & 3e^{3t} \end{pmatrix};$$

at $t = 0$ this is $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = A$.

□

9. Do Problem 26 from Section 6.3.

   Solution. e^{At} is nonsingular because

   (a) its inverse is given by e^{-At},

   (b) its eigenvalues are $e^{\lambda t}$ where λ is an eigenvalue of A — thus 0 is never an eigenvalue of e^{At}.

   $\square$

10. Do Problem 30 from Section 6.3.

   Solution. (a) $\begin{pmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{pmatrix}^{-1} = \frac{1}{1+(\Delta t)^2/4}\begin{pmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{pmatrix}$, so if $\mathbf{U}_n = (Y_n, Z_n)$ we have

$$\mathbf{U}_{n+1} = \frac{1}{1+(\Delta t)^2/4}\begin{pmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{pmatrix}\mathbf{U}_n = A\mathbf{U}_n$$

   where

$$A = \frac{1}{1+(\Delta t)^2/4}\begin{pmatrix} 1-(\Delta t)^2/4 & \Delta t \\ -\Delta t & 1-(\Delta t)^2/4 \end{pmatrix}.$$

   Then

$$\begin{aligned} A^{\mathrm{T}}A &= \frac{1}{(1+(\Delta t)^2/4)^2}\begin{pmatrix} 1-(\Delta t)^2/4 & -\Delta t \\ \Delta t & 1-(\Delta t)^2/4 \end{pmatrix}\begin{pmatrix} 1-(\Delta t)^2/4 & \Delta t \\ -\Delta t & 1-(\Delta t)^2/4 \end{pmatrix} \\ &= \frac{1}{(1+(\Delta t)^2/4)^2}\begin{pmatrix} (1-(\Delta t)^2/4)^2+(\Delta t)^2 & 0 \\ 0 & (1-(\Delta t)^2/4)^2+(\Delta t)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

   If $B^T = -B$ and $A = (I-B)^{-1}(I+B)$ then $A^{\mathrm{T}}A = (I+B^{\mathrm{T}})(I-B^{\mathrm{T}})^{-1}(I-B)^{-1}(I+B) = (I-B)(I+B)^{-1}(I-B)^{-1}(I+B)$. But notice that $(I+B)(I-B) = I-B^2 = (I-B)(I+B)$, hence this equals $(I-B)(I-B)^{-1}(I+B)^{-1}(I+B) = I$. Similarly $AA^{\mathrm{T}} = (I-B)^{-1}(I+B)(I-B)(I+B)^{-1} = (I-B)^{-1}(I-B)(I+B)(I+B)^{-1} = I$, so A is indeed orthogonal.

   (b) If $\Delta t = 2\pi/32$ then using Matlab to compute A^{32} gives

$$\begin{pmatrix} 0.9998 & -0.0201 \\ 0.0201 & 0.9998 \end{pmatrix},$$

   which is close to the identity, but there is clearly a potentially significant error.

   $\square$