18.06 Spring 2012 – Problem Set 3

This problem set is due Thursday, March 1, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

1. Without asking anyone for help, write down an accurate definition of what it means for a matrix to be in reduced row echelon form (RREF).

   Solution. $m \times n$ matrix R is in RREF means

   (a) R is in echelon form.

   (b) Every pivot is 1.

   (c) Columns with a pivot have no other nonzero entry.

   □

2. TRUE or FALSE? (No need for explanation):

   (a) Every upper-triangular matrix is in reduced row echelon form?

   (b) Every lower-triangular matrix is in reduced row echelon form?

   (c) Every permutation matrix is in reduced row echelon form?

   (d) The following matrix is in reduced row echelon form?

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   (e) The reduced row echelon form of A is unique?

   (f) The full solution set of $Ax = b$, where A is $m \times n$ and $b \in \mathbb{R}^m$, is always a vector subspace of $\mathbb{R}^n$?

   (g) The difference $\mathbf{a} = \mathbf{x}_1 - \mathbf{x}_2$, between any two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$, is a vector that belongs to the null space $N(A)$? (Apply the rule $A(\mathbf{x} + \lambda\mathbf{y}) = A\mathbf{x} + \lambda A\mathbf{y}$ to $A(\mathbf{x}_1 - \mathbf{x}_2)$ to answer the question).

   Solution. (a) No. The rows of all zeros must be below all the other rows. This is not true, for instance, of

   $$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   (b) No. For instance,

   $$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

   is not.

(c) No. For instance, for the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the pivot of the second row is to the left of the pivot of the first row.

(d) No. The leading coefficient of the second row is not a one.

(e) Yes. This will be explained in class, though you do not need to know a proof. (The proof-oriented reader should read e.g. http://web.gccaz.edu/ wkehowsk/225-Linear-10-11-Sp/yuster-rref-unique.pdf .)

(f) No. For instance, the solution set of

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

contains a unique vector, $[0, 1]^T$. This is not a vector subspace of $\mathbb{R}^2$.

(g) Yes. $A(\mathbf{x}_1 - \mathbf{x}_2) = A(\mathbf{x}_1) - A(\mathbf{x}_2) = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

□

3. Do Problems 20 & 23 from Section 3.2.

Solution to 3.2.20:

Let A be the matrix in the problem.
The column 5 does not have pivot. If not, since $(A)_{45} = c \neq 0$ is a pivot and $(A)_{4i} = 0$ for any $i \neq 5$, column 1 + column 3 + column 5 = $(*, *, *, c)^T \neq \mathbf{0}$. In other words, the fifth variable x_5 is the only free variable. We have

$$A \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \text{column } 1 \cdot 1 + \text{column } 3 \cdot 1 + \text{column } 5 \cdot 1 = \mathbf{0}.$$

Hence the special solution is $(1, 0, 1, 0, 1)^T$ and the null space is $\{(x_5, 0, x_5, 0, x_5)^T : x_5 \in \mathbf{R}\}$.

Solution to 3.2.23:

$$(a, b, c) = \left(-\frac{1}{2}, -2, -3\right)$$

satisfies the equation

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0.$$

Hence

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

is a matrix we wanted.

4. Do Problem 35 from Section 3.2.

   Solution. The nullspace of $B = [A\,A]$ contains all vectors $\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ -\mathbf{y} \end{bmatrix}$ for all $\mathbf{y}$ in $\mathbb{R}^4$.

   □

5. Do Problems 3 & 8 from Section 3.3.

   Solution to 3.3.3:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = RREF(A).$$

   Using the same elimination or permutation operators as in the case A, we get

$$RREF(B) = [RREF(A) RREF(A)].$$

$$C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} A & A \\ 0 & -A \end{bmatrix} \longrightarrow \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \longrightarrow \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} RREF(A) & 0 \\ 0 & RREF(A) \end{bmatrix} = RREF(C).$$

   Solution to 3.3.8:

   If the matrix has rank 1, every column is constant multiple of any other nonzero columns. So

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}, B = \begin{bmatrix} 3 & 9 & -\frac{9}{2} \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{bmatrix}.$$

   For M, if $a \neq 0$,

$$M = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}$$

   and if $a = 0$,

$$M = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \text{ for any } (b, d) \neq (0, 0), \text{ or } M = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \text{ for any } (c, d) \neq (0, 0).$$

6. Do Problems 17 & 28 from Section 3.3.

   Solution to 3.3.17:

   (a) By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns, then column j of AB is the same combination of earlier columns of AB. Thus rank $(AB) \leq$ rank (B). There are no new pivot columns!

   (b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank of AB stays the same for $A_1 = I$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It drops to zero for $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

   Solution to 3.3.28:

   The row-column echelon form is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; I is the $r \times r$ identity matrix.

7. Do Problems 5 & 16 from Section 3.4.

   Solution to 3.4.5: Consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix}.$$

   The operations those make the first 3×3 matrix to RREF change our augmented matrix to

$$\begin{bmatrix} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{bmatrix}.$$

   Hence this equation is solvable when $-2b_1 - b_2 + b_3 = 0$ and the set of solutions is $\{(5b_1 - 2b_2 + 2z, -2b_1 + b_2, z) : z \in \mathbf{R}\}$.

   Solution to 3.4.16:

   The largest possible rank of a 3 by 5 matrix is 3. Then there is a pivot in every row of U and R. The solution of $Ax = b$ always exists. The column space of A is $\mathbf{R}^3$. An example of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

8. Do Problems 24 & 33 from Section 3.4.

   Solution to 3.4.24:

   (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\mathbf{b}$.

(b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [b]$ has infinitely many solutions for every b.

(c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest examples is a zero matrix.

(d) *One* solution for all $\mathbf{b}$ when A is square and invertible (like $A = I$).

Solution to 3.4.33:

If the complete solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

9. Do Problems 9 from Section 3.5.

Solution. (a) the dimension of $\mathbf{R}^3$ is 3 and 3 is the biggest possible number of independent vectors in $\mathbf{R}^3$.

(b) there exists $(c_1, c_2) \neq (0, 0)$ such that $c_1 \cdot v_1 + c_2 \cdot v_2 = 0$.

(c) $0 \cdot v_1 + 1 \cdot (0, 0, 0) = (0, 0, 0)$.

□

(See Problem 10 on next page!)

10. In this exercise, we try MATLAB's function `null(A)` for finding a basis (i.e. a minimal set of spanning vectors = a maximal set of independent vectors) for the null space of a matrix. We also try `rref(A)` for finding the reduced row echelon form.

```
B = [1  0  0  0;
     0  0  1  0;
     0  0  0  1;
     0  1  0  0];

C = [1  2  1 -2;
     0  0  1  5;
     0  0  0  0;
     0  0  0  0];

D = [1  2  0  1;
     0  2  2  1;
     0  0  3  3;
     1  0  0  4];
```

(a) Using `null()`, find a basis of each of $N(B), N(C)$ and $N(D)$ (the column vectors in the matrix MATLAB outputs are the basis vectors). Same for $N(BC)$ and $N(DC)$.

(b) Figure out whether $N(C)$ and $N(DC)$ are the same subspaces of $\mathbb{R}^4$, as follows: $\longrightarrow$ MATLAB can easily perform this, if we make use of the following two facts, for V and W subspaces of $\mathbb{R}^n$ with given collections of vectors used for spanning them, respectively $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spanning V and $\mathbf{w}_1, \ldots, \mathbf{w}_l$ spanning W.

Fact 1: A vector $\mathbf{b} \in \mathbb{R}^4$ belongs to V *if and only if* the system $A\mathbf{x} = \mathbf{b}$ has at least one solution, where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_k]$ is the matrix which as columns has a collection of vectors we use to span V.

Example (2×2): In MATLAB we create the augmented matrix $[A|\mathbf{b}]$ and use the command `rref`.

```
A = [1 2;
    -1 -2];
b = [1;
     1];

>> A_aug_b = [A b]

A_aug_b =
     1     2  |   1
    -1    -2  |   1
```

```
>> rref(A_aug_b)

ans =
     1     2   |  0
     0     0   |  1
```

(Note: `A_aug_b` is only a variable name. The augmentation bars in the output will not show in MATLAB).

Notice the zero row that has a non-zero entry to the right of the bar: This system $A\mathbf{x} = \mathbf{b}$ has no solution. Hence, $\mathbf{b} = [1, 1]^T$ is not in the subspace spanned by the columns of A.

Fact 2: Two subspaces are the same, $V = W$, *if and only if:*

i. Vectors spanning V lie in W, that is $\mathbf{v}_1, \ldots, \mathbf{v}_k \in W$ (so $V \subseteq W$), and
ii. Vectors spanning W lie in V, that is $\mathbf{w}_1, \ldots, \mathbf{w}_k \in V$ (so $W \subseteq V$).

Example: Referring to the previous example, the subspace V spanned by the vectors $\mathbf{b}$ and $[0, 1]^T$ cannot be the same as the subspace W spanned by the columns of A (since we saw $\mathbf{b} \notin W$).

Now, for using Fact 1 & Fact 2 in MATLAB to determine if $N(C)$ and $N(DC)$ are in fact the same, you will need the ":" option:

```
>> A(:,2)  %Example: Gives you the 2nd column from matrix A
```

Then proceed as in the examples, checking each basis vector from one space for membership of the other space.

(c) Which property of the square matrix D explains the result of your comparison of $N(C)$ and $N(DC)$? State this as a general rule, and put a box around it. Apply your rule to explain why $N(DC)$ and $N(BC)$ are the same subspace.

(d) Is $N(CB)$ the same as $N(C)$? Either use the method from (b) again (you can do it all at once using `rref([null(CB) null(C)])`, if you carefully read off the result!), or simply try applying CB to the basis vectors you found for $N(C)$, and vice versa.

Solution. (a) Bases for the null spaces are as follows:

```
>> null(B)

ans =

   Empty matrix: 4-by-0

>> null(C)

ans =

         0    0.9245
    0.5659   -0.3142
```

```
   -0.8085   -0.2115
    0.1617    0.0423

>> null(D)

ans =

   Empty matrix: 4-by-0

>> null(B*C)

ans =

         0   -0.9245
    0.5659    0.3142
   -0.8085    0.2115
    0.1617   -0.0423

>> null(D*C)

ans =

   -0.0331    0.9239
   -0.5543   -0.3343
    0.8155   -0.1824
   -0.1631    0.0365
```

(b) Here we get, for example:

```
nC=null(C);
nC_1 = nC(:,1);
nC_2 = nC(:,2);

nDC = null(D*C);

redux1 = rref([nDC nC_1])
redux2 = rref([nDC nC_2])

>> redux1 =

    1.0000         0   | -0.9994
         0    1.0000   | -0.0358
         0         0   |      0
         0         0   |      0
```

```
redux2 =

    1.0000         0    | -0.0358
         0    1.0000    |  0.9994
         0         0    |     0
         0         0    |     0
```

Here we looked at the system $A\mathbf{x} = \mathbf{b}_i$, with A's columns being the basis of $N(DC)$ and for $i = 1, 2$ let $\mathbf{b}_1 =$ `nC_1`, $\mathbf{b}_2 =$ `nC_1` be the two basis vectors we got for $N(C)$ in (a). Since in both cases the system is consistent (the zero rows in the left compartment of the above RREF of the augmented matrix has a corresponding zero in the right compartment).

Thus, since $\mathbf{b_1}, \mathbf{b_2} \in N(DC)$ and since $N(C)$ is spanned by its two basis vectors $\mathbf{b_1}, \mathbf{b_2}$, we conclude that: $N(C) \subseteq N(DC)$.

Note: $N(C) \subseteq N(DC)$ is true for *any* matrices D and C! Why? Because if $\mathbf{x} \in N(C)$, meaning $C\mathbf{x} = 0$, then also $DC\mathbf{x} = D\mathbf{0} = \mathbf{0}$ meaning $\mathbf{x} \in N(DC)$.

Similar code, reversing the roles of C and DC checks for us that also (which is not always true - see below): $N(DC) \subseteq N(C)$.

Thus, we have checked that: $N(DC) = N(C)$.

(c) The property the square matrix D has is: Invertible. Here's the rule:

If D, C are any $n \times n$ matrices, and D invertible, then $N(DC) = N(C)$.

We saw the invertibility of D above in (a): The basis for the null space was $\emptyset$ (the empty set), so $N(D) = \{0\}$ (the subspace only consisting of the zero vector). Thus, if we reduced D to its RREF matrix R we would obtain the 4×4 identity I (since D is square!). But this means that D is invertible.

This also explains why B is invertible, using (a). Now, we may use our new rule: $N(DC) = N(D(B^{-1}B)C) = N((DB^{-1})BC) = N(BC)$.

(d) No, $N(CB)$ and $N(C)$ are not the same (in this example).

```
nC=null(C);
nCB=null(C*B);
BigMat = rref([nCB nC]);

BigMat =

     1     0  |   0     0
     0     1  |   0     0
     0     0  |   1     0
     0     0  |   0     1
```

Note that we have solved all the four systems at once by using the augmentation. Reading left-to-right, you can see that none of the two basis vectors MATLAB chose for us for $N(C)$ belong to $N(CB)$. Reading right-to-left, we see that reversely the $N(CB)$ basis we chose is not in $N(C)$. So these subspaces are not identical.

Alternatively, we can try:

```
C*B*nC_1

ans =

   -2.5870
    2.9913
         0
         0
```

Since that's not the zero vector, we have that the vector `nC_1` from $N(C)$ is not in $N(CB)$. So, these two subspaces are not the same.

□