18.06 Spring 2012 – Problem Set 1 - Solutions

This problem set is due Thursday, February 16th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 8 from Section 1.3.

   Solution to 1.3.8:

$$\begin{aligned} x_1 - 0 &= b_1 & x_1 &= b_1 \\ x_2 - x_1 &= b_2 & x_2 &= b_1 + b_2 \\ x_3 - x_2 &= b_3 & x_3 &= b_1 + b_2 + b_3 \\ x_4 - x_3 &= b_4 & x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\mathbf{b}$$

2. Do Problem 8 & Problem 32 from Section 2.2.

   Solution to 2.2.8:

   If $k = 3$, then elimination must fail: No solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: Infinitely many solutions. If $k = 0$ a row exchange is needed: Exactly one solution.

   Solution to 2.2.32:

   The question deals with 100 equations $Ax = 0$ when A is singular.

   (a) Some linear combination of the 100 rows is the row of 100 zeros.

   (b) Some linear combination of the 100 columns is the column of zeros.

   (c) A very singular matrix has all ones: $A = \mathbf{eye}(100)$. A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

   (d) The row picture has 100 planes meeting along a common line through 0. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

3. Do Problem 22 from Section 2.3.

   Solution to 2.3.22:

   (a) $\sum a_{3j}x_j$.

   (b) $a_{21} - a_{11}$.

(c) $a_{21} - 2a_{11}$.

(d) $(EAx)_1 = (Ax)_1 = \sum_j a_{1j}x_j$.

4. Do Problem 19 & Problem 36 from Section 2.4.

Solution to 2.4.19:

(a) a_{11}.

(b) $l_{31} = a_{31}/a_{11}$.

(c) $a_{32} - \left(\frac{a_{31}}{a_{11}}\right) a_{12}$.

(d) $a_{22} - \left(\frac{a_{21}}{a_{11}}\right) a_{12}$.

Solution to 2.4.36:

Multiplying $AB = (m \text{ by } n)(n \text{ by } p)$ needs mnp multiplications. Then $(AB)C$ needs mpq more. Multiply $BC = (n \text{ by } p)(p \text{ by } q)$ needs npq and then A(BC) needs mnq.

(a) If m, n, p, q are $2, 4, 7, 10$ we compare $(2)(4)(7)+(2)(7)(10) = 196$ with the larger number $(2)(4)(10)+(4)(7)(10) = 360$. So AB first is better, so that we multiply that 7 by 10 matrix by as few rows as possible.

(b) If u, v, w are N by 1, then $(u^T v)w^T$ needs $2N$ multiplications but $u^T(vw^T)$ needs N^2 to find vw^T and N^2 more to multiply by the row vector u^T. Apologies to use the transpose symbol so early.

(c) We are comparing $mnp + mpq$ with $mnq + npq$. Divide all terms by $mnpq$: Now we are comparing $q^{-1}+n^{-1}$ with $p^{-1}+m^{-1}$. This yields a simple important rule. If matrices A and B are multiplying v for ABv, don't multiply the matrices first.

5. For which values of q (if any) is the following system consistent (= solvable)?

$$x + 4y + 3z = 1,$$
$$q^3x + 4q^3y + 3q^3z = 64q.$$

Solution: We write the system as a matrix equation

$$\begin{bmatrix} 1 & 4 & 3 \\ q^3 & 4q^3 & 3q^3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 64q \end{bmatrix}.$$

In a one-step elimination, we get for the augmented matrix $[A \mid b]$:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 0 & 0 & 64q - q^3 \end{array}\right]$$

The equation $0 = 64q - q^3 = q(64 - q^2)$ holds if either $q = 0$ or $64 - q^2 = 0$, so:

$$\boxed{\text{Only consistent when either } q = 0,\ q = -8 \text{ or } q = 8.}$$

6. A permutation matrix P comes from permuting the rows of the identity matrix I_n. If the entries of P are labelled p_{ij}, the matrix A having entries $a_{ij} = p_{ji}$ is the transpose, $A = P^T$.

   (a) Is P invertible, and if yes *why*? How would we proceed in Gaussian elimination on P?

   (b) Explain why the product $C = PP^T$ is the identity matrix. Think about where the 1's and 0's are.

   (c) Since the answer to (a) was "yes", what is the inverse to P?

   Solution:

   (a) Yes. To proceed we would swap all rows back in their correct place and obtain the identity. Hence P is invertible.

   (b) Look at the entry c_{ij} in C, which is the dot product of the i'th row in P and the j'th column of P^T, the latter of which is simply the j'th row of P.
   For the identity matrix, each row dotted with itself gives 1, while no two (different) rows have a non-zero dot product - these properties are not changed when we swap the rows, so c_{ij} is 1 when $i = j$, and zero whenever $i \neq j$. So, we see $C = I$.

   (c) Using (b), we see $P^{-1} = P^T$.

   Note: This exercise says a permutation matrix is *orthogonal*: $PP^T = P^T P = I$.

7. (a) Give examples of non-zero (meaning: not all entries zero) 2×2 and 4×4 matrices A, one of each, such that $A^2 = O$ (recall O means the zero matrix). Hint: You only need to use one 1, and the rest of the entries can be 0's!

   (b) Are there any invertible $n \times n$ matrices A such that $A^2 = O$?

   Solution:

   (a) In both cases, putting a 1 in the top right corner and the rest of the entries to 0 works.

   (b) No. Since then $A = A^{-1}A^2 = A^{-1}O = O$.

8. Given the three vectors $\mathbf{a}_1 = (1, 2, 3)$, $\mathbf{a}_2 = (1, 0, -1)$ and $\mathbf{a}_3 = (0, 0, 1)$, find (if possible) numbers x_1, x_2 and x_3 such that:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   Your solution should involve Gaussian elimination on $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ (the matrix with $\mathbf{a}_i$'s as columns).

   Solution:

   The answer is: $x_1 = 1/2$, $x_2 = 1/2$ and $x_3 = 0$.

9. (a) Using MATLAB, perform the matrix products A^2, A^3 and A^6 of the following lower-triangular matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 2 & 0 & 0 \\ 5 & 1 & 3 & 0 \\ 3 & 2 & -1 & 4 \end{bmatrix}$$

(b) Explain the rule for *diagonal* entries of A^k, for a lower-triangular matrix A.

(c) Guess a rule for the $(2,1)$ entry of A^k, for a lower-triangular matrix A.

Solution:

(a) The MATLAB output looks like this:

```
>> A^2

ans =

     1     0     0     0
    21     4     0     0
    27     5     9     0
    24    11    -7    16

>> A^3

ans =

     1     0     0     0
    49     8     0     0
   107    19    27     0
   114    47   -37    64

>> A^6

ans =

           1           0           0           0
         441          64           0           0
        3927         665         729           0
        5754        2681       -3367        4096
```

(b) For a lower-triangular (or upper-) matrix A, the rule

$$(A^k)_{ii} = (a_{ii})^k$$

holds.

(c) Deriving is maybe better than guessing? Let us for brevity write $b_k = (A^k)_{21}$. Hence $b_1 = a_{21}$. Since $A^k = AA^{k-1}$ we compute that:

$$b_k = (A^k)_{21} = a_{21}a_{11}^{k-1} + a_{22}(A^{k-1})_{21} = b_1 a_{11}^{k-1} + a_{22}b_{k-1}.$$

Baby case. If we had $a_{22} = 1$, we could more easily see what would happen:

$$b_k = b_{k-1} + b_1 a_{11}^{k-1}.$$

Thus we have $b_3 = b_2 + b_1 a_{11}^2 = b_1 + b_1 a_{11} + b_1 a_{11}^2$ and so on, leading to:

$$(A^k)_{21} = b_k = b_1 \sum_{s=0}^{k-1} a_{11}^s = b_1 \frac{1 - a_{11}^k}{1 - a_{11}} = a_{21} \frac{1 - a_{11}^k}{1 - a_{11}}.$$

In the second-to-last equality we used the sum formula for a finite geometric series, valid when $a_{11} \neq 1$ (we leave the case $a_{11} = 1$ to the reader!).

General case. Note that we can reduce to the special case by scaling: We let $C = \frac{1}{a_{22}} A$ (and leave the special case $a_{22} = 0$ to the reader!). Then, using our formula above (that works since $c_{21} = 1$) we see:

$$(A^k)_{21} = a_{22}^k (C^k)_{21} = a_{22}^k c_{21} \frac{1 - c_{11}^k}{1 - c_{11}} = a_{22}^{k-1} a_{21} \frac{1 - (\frac{a_{11}}{a_{22}})^k}{1 - \frac{a_{11}}{a_{22}}}.$$

Thus, we finally see:

$$\boxed{(A^k)_{21} = a_{21} \frac{a_{22}^k - a_{11}^k}{a_{22} - a_{11}} \quad (\text{when} \quad a_{11} \neq a_{22})}$$

CHECK: For example, in the above MATLAB output,

$$(A^6)_{21} = 7 \frac{2^6 - 1^6}{2 - 1} = 441. \quad \checkmark$$

10. A chemistry professor claimed on live TV that he could, by mixing, obtain *any* wine with given contents of water (W), sugar (S) and tannic acid (T), labelled by vectors $w = (W, S, T)$ such that $W + S + T = 100\%$. Due to a lack of research funding, his stock was quite limited:

- Laboratory water supply: $w_1 = (100, 0, 0)$.
- Budget wine: $w_2 = (50, 0, 50)$.
- Plum tea concentrate: $w_3 = (30, 50, 20)$.

(a) If a Chateaux Bordeaux 1915 has $(W, S, T) = (45, 50, 5)$, why was the professor *not* able to obtain this wine by mixing w_1, w_2, w_3? Explain by computing the mixing ratios needed (by MATLAB or by hand).

(b) Help the professor restore honor, by adding any new wine w_4 that will enable him to make the Chateaux Bordeaux 1915 (a Chateaux Bordeaux 1915 not allowed!).

(c) Are the mixing ratios unique after addition of the fourth wine?

Solution:

(a) The result is $(W, S, T) = (3/10, -3/10, 1)$. Since you would need to be able to *subtract* an amount of *Plum tea concentrate*, which is physically intractable, there is no mixing that will work.

(b) We can for example pick $w_4 = (40, 60, 0)$ (note that it sums to 100%, hence is an admissible wine).

The wine matrix $A = [w_1 \quad w_2 \quad w_3 \quad w_4]$ then reads:

$$A = \begin{bmatrix} 100 & 50 & 30 & 40 \\ 0 & 0 & 50 & 60 \\ 0 & 50 & 20 & 0 \end{bmatrix}.$$

But we can now also forget entirely about, say, the second wine w_2 (see the Figure 1 on the last page of these solutions), and consider instead the square matrix $A_2 = [w_1 \quad w_3 \quad w_4]$ which is:

$$A_2 = \begin{bmatrix} 100 & 30 & 40 \\ 0 & 50 & 60 \\ 0 & 20 & 0 \end{bmatrix}.$$

Using Gauss elimination on $[A_2 \mid \mathbf{b}]$ to solve $A_2\mathbf{x}_2 = \mathbf{b}$, where $\mathbf{b} = (45, 50, 5)$, we find:

$$\mathbf{x}_2 = \begin{bmatrix} 1/8 \\ 1/4 \\ 5/8 \end{bmatrix}.$$

Note that all the solution's entries automatically sum to 1.

(c) No - in this situation, $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions, and also infinitely many solutions that are admissible (i.e. have positive entries).

Later, after a few more weeks of 18.06, you will know how to obtain the complete solution to $A\mathbf{x} = \mathbf{b}$. We record it here for insight, and later reference:

$$\mathbf{x} = \begin{bmatrix} 1/8 \\ 0 \\ 1/4 \\ 5/8 \end{bmatrix} + s \begin{bmatrix} -7/12 \\ 1 \\ -5/2 \\ 25/12 \end{bmatrix}, \quad s \in \mathbb{R}.$$

Note that all these sum to 100%. Here we can in fact pick any s in the interval $s \in [0, 3/14]$ and still have non-negative entries in $\mathbf{x}$.

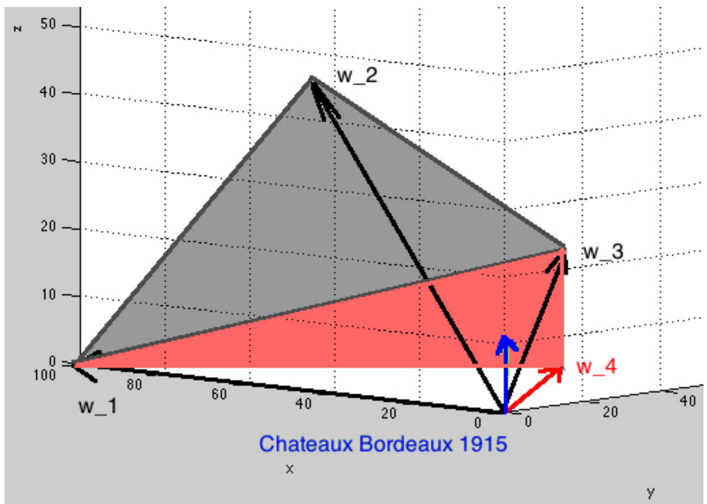

Figure 1: Problem 10. The grey and salmon-colored triangles are subsets of the plane $x + y + z = 100$ (i.e. admissible wines) with only positive mixing amounts of the w_i's.