18.06 Professor Strang Final Solutions May 23rd, 2012

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

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1 (12 pts.)

(a) - Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

Solution. The eigenvalues are:

$$\boxed{\lambda = 0, 3, 6}$$

The corresponding eigenvectors are:

$$\boxed{\lambda = 0: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix}}$$

$$\boxed{\lambda = 3: \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\boxed{\lambda = 6: \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}}$$

□

(b) - Write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of eigenvectors of A.

- Find the vector $A^{10}\mathbf{v}$.

Solution. We have that, forming $T = [\, v_1 \mid v_2 \mid v_3 \,]$ (with columns = the three vectors),

$$\boxed{\mathbf{y} = T^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \end{bmatrix}}$$

Or in other words:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

Therefore, we also see:

$$A^{10}\mathbf{v} = -3^{10}\frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6^{10}\frac{1}{3} \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} 100737594 \\ 60466176 \\ 60466176 \end{bmatrix}$$

(*) Required for mental arithmetics wizards only. □

(c) If you solve $\frac{d\mathbf{u}}{dt} = -A\mathbf{u}$ (notice the minus sign), with $\mathbf{u}(0)$ a given vector, then as $t \to \infty$ the solution $\mathbf{u}(t)$ will always approach a multiple of a certain vector $\mathbf{w}$.

- Find this steady-state vector $\mathbf{w}$.

Solution. Since the eigenvalues of $-A$ are $0, -3, -6$, we see that this steady state is:

$$\mathbf{w} = v_1 = \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix}$$

□

2 (12 pts.)

Suppose A has rank 1, and B has rank 2 (A and B are both 3×3 matrices).

(a) - What are the possible ranks of $A + B$?

Solution. Of course, $0 \le \text{rank}(A + B) \le 3$. But the only ranks that are possible are:

$$\boxed{\text{rank}(A + B) = 1, 2, 3.}$$

The reason 0 is not an option is: It implies $A + B = 0$, i.e. that $A = -B$. But $\text{rank}(-B) = \text{rank}(B)$, so for that to happen A and B should have had the same rank. □

(b) - Give an example of each possibility you had in (a).

Solution. Here are some simple examples:

Example w/ $\text{rank}(A + B) = 1$: Take e.g.

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Example w/ $\text{rank}(A + B) = 2$: Take e.g.

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Example w/ $\text{rank}(A + B) = 3$: Take e.g.

$$\boxed{A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

□

(c) - What are the possible ranks of AB?

- Give an example of each possibility.

Solution. As a general rule, recall $0 \leq \text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) = 1$. In this case, both possibilities do happen:

$$\boxed{\text{rank}(AB) = 0, 1.}$$

Diagonal examples suffice:

<u>Example w/ rank$(AB) = 0$</u>: Take e.g.

$$\boxed{A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

<u>Example w/ rank$(AB) = 1$</u>: Take e.g.

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

□

3 (12 pts.)

(a) - Find the three pivots and the determinant of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Solution. We see that

$$\boxed{A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}}$$

Thus,

$$\boxed{\text{The pivots are } 1, 1, -2}$$

Since we reduced A without any *row switches* (permutation P's), or row scalings, we have:

$$\boxed{\det A = 1 \cdot 1 \cdot (-2) = -2}$$

□

(b) - The rank of $A - I$ is ______, so that $\lambda =$______ is an eigenvalue.

- The remaining two eigenvalues of A are $\lambda =$____________.

- These eigenvalues are all ________________________, because $A^T = A$.

Solution. We see that

$$\boxed{\text{rank}(A - I) = 2}$$

So $\dim N(A - I) = 1$. Thus,

$$\boxed{\lambda = 1}$$

is an eigenvalue of algebraic and geometric multiplicity one.

The other two eigenvalues of A are:

$$\boxed{\lambda = -1, 2.}$$

$$\boxed{\text{The eigenvalues are all real values, because } A \text{ is symmetric.}}$$

$\square$

(c) The unit eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ will be orthonormal.

- Prove that:

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T.$$

You may compute the $\mathbf{x}_i$'s and use numbers. Or, without numbers, you may show that the right side has the correct eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Solution. As suggested, we check that A does the correct thing on the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2\}$.

$$\boxed{\left(\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T\right) \mathbf{x_i} = \lambda_i (\mathbf{x}_i^T \mathbf{x}_i) \mathbf{x}_i = \lambda_i \mathbf{x}_i = A \mathbf{x}_i}$$

Having checked this, then by linearity of matrix multiplication, the two expressions agree always (and hence the matrices are identical).

For the record, the three vectors are:

$$\mathbf{x_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{x_3} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$\square$

4 (12 pts.)

This problem is about $x + 2y + 2z = 0$, which is the equation of a plane through $\mathbf{0}$ in $\mathbb{R}^3$.

(a) - That plane is the nullspace of what matrix A?

$A =$

- Find an orthonormal basis for that nullspace (that plane).

Solution.

$$\boxed{A = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}}$$

□

We could identify a basis of $N(A)$ as usual, then apply Gram-Schmidt to make it an orthonormal basis.

But if we can find two orthonormal vectors in $N(A)$, we are done. Here, one can first easily guess one vector in $N(A)$:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in N(A)$$

Then anything of the form $[a \quad 1 \quad 1]$ will be orthogonal to $\mathbf{v}_1$, and we pick the one that is in the null space:

$$\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \in N(A)$$

Then an orthonormal basis is:

$$\boxed{\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}}$$

(b) That plane is the column space of many matrices B.

- Give two examples of B.

Solution. We can use the basis vectors from above as columns, and (independent) linear combinations of them. Or filling in a zero column:

$$\boxed{B_1 = [\mathbf{v}_1 \quad \mathbf{v}_2]}$$

$$\boxed{B_2 = [\mathbf{v}_1 \quad 2\mathbf{v}_1 + \mathbf{v}_2]}$$

$$\boxed{B_3 = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{0}]}$$

□

Then $c(B_i) = N(A)$.

(c) - How would you compute the projection matrix P onto that plane? (A formula is enough)

- What is the rank of P?

Solution. It can be computed using a matrix B from above (if it has *independent* columns: So B_1, B_2 but not B_3 here), via the usual formula:

$$\boxed{P = B(B^T B)^{-1} B^T}$$

For a projection, $c(P)$ is always the subpace it projects on, in this case it is the two-dimensional plane:

$$\boxed{\text{rank}(P) = \dim c(P) = 2}$$

□

5 (12 pts.)

Suppose $\mathbf{v}$ is any unit vector in $\mathbb{R}^3$. This question is about the matrix H.

$$H = I - 2\mathbf{v}\mathbf{v}^T.$$

(a) - Multiply H times H to show that $H^2 = I$.

Solution.

$$\boxed{H^2 = (I - 2\mathbf{v}\mathbf{v}^T)^2 = I^2 + 4(\mathbf{v}\mathbf{v}^T)^2 - 4\mathbf{v}\mathbf{v}^T = I + 4\mathbf{v}\mathbf{v}^T - 4\mathbf{v}\mathbf{v}^T = I}$$

□

(b) - Show that H passes the tests for being a symmetric matrix <u>and</u> an orthogonal matrix.

Solution. Transpose is linear, $I^T = I$, and anything of the form AA^T is symmetric:

$$\boxed{(I - 2\mathbf{v}\mathbf{v}^T)^T = I - 2(\mathbf{v}^T)^T\mathbf{v}^T = I - 2\mathbf{v}\mathbf{v}^T}$$

For orthogonality, we use (a) and symmetry:

$$\boxed{HH^T = H^2 = I}$$

□

(c) - What are the eigenvalues of H?

You have enough information to answer for any unit vector $\mathbf{v}$, but you can choose one $\mathbf{v}$ and compute the λ's.

Solution. Note first that (since $\|\mathbf{v}\| = 1$):

$$H\mathbf{v} = \mathbf{v} - 2(\mathbf{v}^T\mathbf{v})\mathbf{v} = -\mathbf{v},$$

so that

$$\boxed{\lambda = -1}$$

is an eigenvalue (with a one-dimensional eigenspace spanned by $\mathbf{v}$).

Let on the other hand $\mathbf{u} \in \left(\text{span}\{\mathbf{v}\}\right)^\perp$ be any vector orthogonal to $\mathbf{v}$. Then we have:

$$H\mathbf{u} = \mathbf{u} - 2(\mathbf{v}^T\mathbf{u})\mathbf{v} = \mathbf{u},$$

so that

$$\boxed{\lambda = 1}$$

is also an eigenvalue.

Since $\left(\text{span}\{\mathbf{v}\}\right)^{\perp}$ is two-dimensional, we have found all eigenvalues.

□

6 (12 pts.)

(a) - Find the closest straight line $y = Ct + D$ to the 5 points:

$$(t, y) = (-2, 0),\ \ (-1, 0),\ \ (0, 1),\ \ (1, 1),\ \ (2, 1).$$

Solution. We insert all points into the equation:

$$\begin{aligned} -2C + D &= 0 \\ -C + D &= 0 \\ 0 + D &= 1 \\ 1 + D &= 1 \\ 2C + D &= 1. \end{aligned}$$

Written as a matrix system:

$$A\mathbf{x} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{b}$$

We consider instead $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. We compute:

$$A^T A = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}, \qquad A^T\mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

and

$$(A^T A)^{-1} = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix}.$$

Thus finally:

$$\begin{bmatrix} C \\ D \end{bmatrix} = \hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b} = \begin{bmatrix} 3/10 \\ 3/5 \end{bmatrix}.$$

So, the closest line to the five points is:

$$\boxed{y = \frac{3}{10}t + \frac{3}{5}.}$$

□

(b) - The word "closest" means that you minimized which quantity to find your line?

Solution. It means that the sum of squares deviation $\|A\mathbf{x} - \mathbf{b}\|^2$ was minimized.

□

(c) If $A^T A$ is invertible, what do you know about its eigenvalues and eigenvectors? (Technical point: Assume that the eigenvalues are distinct – no eigenvalues are repeated). Since $A^T A$ is symmetric and $\mathbf{x} \cdot (A^T A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$ always, it is positive semi-definite. Since $N(A^T A) = \{0\}$, zero is not eigenvalue. Hence:

$$\boxed{\text{The eigenvalues of } A^T A \text{ are positive, if } A^T \text{ is invertible}}$$

By symmetry:

$$\boxed{\text{Eigenvectors belonging to different eigenvalues are orthogonal}}$$

7 (12 pts.)

This symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \text{and} \quad H^2 = 4I.$$

(a) What is the determinant of H?

Solution. By row reduction, we get the pivots $1, -2, -2, 4$, so:

$$\boxed{\det H = 16}$$

□

(b) What are the eigenvalues of H? (Use $H^2 = 4I$ and the trace of H).

Solution. By $H^2 = 4I$, the eigenvalues are all either ± 2. They sum up to $\text{tr}H = 0$. Hence:

$$\boxed{\text{Two eigenvalues must be } +2\text{, and two eigenvalues be } -2}$$

Note also that this shows $\det H = 16$ as in (a) □

(c) What are the singular values of H?

$$\boxed{\text{The singular values of } H \text{ are } 2, 2, 2, 2}$$

8 (16 pts.)

In this TRUE/FALSE problem, you should *circle* your answer to each question.

(a) Suppose you have 101 vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{101} \in \mathbb{R}^{100}$.

- Each v_i is a combination of the other 100 vectors: TRUE – (FALSE)

- Three of the v_i's are in the same 2-dimensional plane: TRUE – (FALSE)

(b) Suppose a matrix A has repeated eigenvalues $7, 7, 7$, so $\det(A - \lambda I) = (7 - \lambda)^3$.

- Then A certainly cannot be diagonalized ($A = S\Lambda S^{-1}$): TRUE – (FALSE)

- The Jordan form of A must be $\mathcal{J} = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$: TRUE – (FALSE)

(c) Suppose A and B are 3×5.

- Then $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$: (TRUE) – FALSE

(d) Suppose A and B are 4×4.

- Then $\det(A+B) \leq \det(A) + \det(B)$: TRUE – (FALSE)

(e) Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthonormal, and call the vector $\mathbf{b} = 3\mathbf{u} + \mathbf{v}$. Take V to be the line of all multiples of $\mathbf{u} + \mathbf{v}$.

- The orthogonal projection of $\mathbf{b}$ onto V is $2\mathbf{u} + 2\mathbf{v}$: (TRUE) – FALSE

(f) Consider the transformation $T(x) = \int_{-x}^{x} f(t)dt$, for a fixed function f. The input is x, the output is $T(x)$.

- Then T is always a linear transformation: TRUE – (FALSE)