18.06 Solutions to PSet 9

6.7:

3: If A has rank 1 then so does $A^{\mathrm{T}}A$. The only nonzero eigenvalue of $A^{\mathrm{T}}A$ is its trace, which is the sum of all a_{ij}^2. (Each diagonal entry of $A^{\mathrm{T}}A$ is the sum of a_{ij}^2 down one column, so the trace is the sum down all columns.) Then $\sigma_1 =$ square root of this sum, and $\sigma_1^2 =$ this sum of all a_{ij}^2.

6: $AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $\boldsymbol{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\sigma_2^2 = 1$ with $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$;

and $\boldsymbol{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [\,\boldsymbol{u}_1 \;\; \boldsymbol{u}_2\,] \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [\,\boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \boldsymbol{v}_3\,]^{\mathrm{T}}$.

7: The matrix A in Problem 6 had $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$ in Σ. The smallest change to rank 1 is **to make $\boldsymbol{\sigma}_2 = \mathbf{0}$**. In the factorization

$$A = U\Sigma V^{\mathrm{T}} = \boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\sigma_2\boldsymbol{v}_2^{\mathrm{T}}$$

this change $\sigma_2 \to 0$ will leave the closest rank–1 matrix as $\boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}}$. See Problem 14 for the general case of this problem.

9: $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

10: A rank–1 matrix with $A\boldsymbol{v} = 12\boldsymbol{u}$ would have $\boldsymbol{u}$ in its column space, so $A = \boldsymbol{u}\boldsymbol{w}^{\mathrm{T}}$ for some vector $\boldsymbol{w}$. I intended (but didn't say) that $\boldsymbol{w}$ is a multiple of the unit vector $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ in the problem. Then $A = 12\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ to get $A\boldsymbol{v} = 12\boldsymbol{u}$ when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = 1$.

11: If A has orthogonal columns $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ of lengths $\sigma_1, \ldots, \sigma_n$, then $A^{\mathrm{T}}A$ will be diagonal with entries $\sigma_1^2, \ldots, \sigma_n^2$. So the σ's are definitely the singular values of A (as expected). The eigenvalues of that diagonal matrix $A^{\mathrm{T}}A$ are the columns of I, so $V = I$ in the SVD. Then the $\boldsymbol{u}_i$ are $A\boldsymbol{v}_i/\sigma_i$ which is the unit vector $\boldsymbol{w}_i/\sigma_i$.

The SVD of this A with orthogonal columns is $A = U\Sigma V^{\mathrm{T}} = (A\Sigma^{-1})(\Sigma)(I)$.

14: he smallest change in A is to set its smallest singular value σ_2 to zero. See # **7**.

15: The singular values of $A + I$ are not $\sigma_j + 1$. They come from eigenvalues of $(A+I)^{\mathrm{T}}(A+I)$.

8.1:

3: The rows of the free-free matrix in equation (9) add to $[\,0 \;\; 0 \;\; 0\,]$ so the right side needs $f_1 + f_2 + f_3 = 0$. $\boldsymbol{f} = (-1,0,1)$ gives $c_2u_1 - c_2u_2 = -1, c_3u_2 - c_3u_3 = -1, 0 = 0$. Then $\boldsymbol{u}_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}} = (1,1,1)$.

4: $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx = -\left[c(x)\frac{du}{dx}\right]_0^1 = 0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx = 0$.

7: For 5 springs and 4 masses, the 5 by 4 A has two nonzero diagonals: all $a_{ii} = 1$ and $a_{i+1,i} = -1$. With $C = \mathsf{diag}(c_1, c_2, c_3, c_4, c_5)$ we get $K = A^{\mathrm{T}}CA$, symmetric tridiagonal with diagonal entries $K_{ii} = c_i + c_{i+1}$ and off-diagonals $K_{i+1,i} = -c_{i+1}$. With $C = I$ this K is the $-1, 2, -1$ matrix and $K(2, 3, 3, 2) = (1, 1, 1, 1)$ solves $K\boldsymbol{u} = \mathsf{ones}(4, 1)$. ($K^{-1}$ will solve $K\boldsymbol{u} = \mathsf{ones}(4)$.)