18.06 Solutions to PSet 8

6.4:

5: $Q = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of A. Each unit eigenvector could be multiplied by -1

7: (a) (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots have the same signs as the λ's (c) trace $= \lambda_1 + \lambda_2 = 2$, so A can't have two negative eigenvalues. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots have the same signs as the λ's (c) trace $= \lambda_1 + \lambda_2 = 2$, so A can't have two negative eigenvalues.

16: (a) If $A\boldsymbol{z} = \lambda\boldsymbol{y}$ and $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{z}$ then $B[\,\boldsymbol{y};\ -\boldsymbol{z}\,] = [\,-A\boldsymbol{z};\ A^{\mathrm{T}}\boldsymbol{y}\,] = -\lambda[\,\boldsymbol{y};\ -\boldsymbol{z}\,]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}A\boldsymbol{z} = A^{\mathrm{T}}(\lambda\boldsymbol{y}) = \lambda^2\boldsymbol{z}$. (c) $\lambda = -1, -1, 1, 1$; $\boldsymbol{x}_1 = (1, 0, -1, 0)$, $\boldsymbol{x}_2 = (0, 1, 0, -1)$, $\boldsymbol{x}_3 = (1, 0, 1, 0)$, $\boldsymbol{x}_4 = (0, 1, 0, 1)$.

23: A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows $QR, S\Lambda S^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

6.5:

8: $A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Pivots $3, 4$ outside squares, ℓ_{ij} inside. $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 3(x + 2y)^2 + 4y^2$

12: A is positive definite for $c > 1$; determinants $c, c^2 - 1$, and $(c-1)^2(c+2) > 0$. B is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

19: All cross terms are $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$ because symmetric matrices have orthogonal eigenvectors. So positive eigenvalues $\Rightarrow$ positive energy.

20: (a) The determinant is positive; all $\lambda > 0$ (b) All projection matrices except I are singular (c) The diagonal entries of D are its eigenvalues (d) $A = -I$ has $\det = +1$ when n is even.

22: $R = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix}\dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $R = Q\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

26: The Cholesky factors $C = \left(L\sqrt{D}\right)^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D. Note again $C^{\mathrm{T}}C = LDL^{\mathrm{T}} = A$.

33: A product AB of symmetric positive definite matrices comes into many applications. The "*generalized*" eigenvalue problem $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ has $AB = M^{-1}K$. (often we use *eig*(K, M) without actually inverting M.) All eigenvalues λ are positive:

$$AB\boldsymbol{x} = \lambda\boldsymbol{x} \text{ gives } (B\boldsymbol{x})^{\mathrm{T}}AB\boldsymbol{x} = (B\boldsymbol{x})^{\mathrm{T}}\lambda x. \text{ Then } \lambda = \boldsymbol{x}^{\mathrm{T}}B^{\mathrm{T}}AB\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}B\boldsymbol{x} > 0.$$

6.6:

2: $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1}AM$ with $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are similar (they all have eigenvalues 1 and 0). $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is by itself and also $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is by itself with eigenvalues 1 and -1.

8: Same Λ Same S But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

12: If $M^{-1}JM = K$ then $JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & \mathbf{0} & \mathbf{0} & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} \mathbf{0} & m_{12} & m_{13} & \mathbf{0} \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$.

That means $m_{21} = m_{22} = m_{23} = m_{24} = 0$. M is not invertible, J not similar to K.

21: J^2 has three 1's down the *second* superdiagonal, and *two* independent eigenvectors for $\lambda = 0$. Its 5 by 5 Jordan form is $\begin{bmatrix} J_3 & \\ & J_2 \end{bmatrix}$ with $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.