18.06 Solutions to PSet 7

8.3:

3: $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; 1 and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
4: A^{T} always has the eigenvector $(1, 1, \ldots, 1)$ for $\lambda = 1$, because each row of A^{T} adds to 1. (Note again that many authors use row vectors multiplying Markov matrices. So they transpose our form of A.)
7: $(.5)^k \to 0$ gives $A^k \to A^{\infty}$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \le 1 \\ .4 + .6a \ge 0 \end{matrix}$
12: B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3,\ .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t} \boldsymbol{x}_1 = c_1 \boldsymbol{x}_1$.
15: The first two A's have $\lambda_{\max} < 1$; $\boldsymbol{p} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.
16: $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state $(.3, .3, .4)$ and $(30, 30, 40)$.

6.3:

4: $d(v + w)/dt = (w - v) + (v - w) = 0$, so the total $v + w$ is constant. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -2 \end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\begin{matrix} v(1) = 20 + 10e^{-2} & v(\infty) = 20 \\ w(1) = 20 - 10e^{-2} & w(\infty) = 20 \end{matrix}$
7: A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (1, 1)$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (1, -1)$. For the solution to $\boldsymbol{u}' = -P\boldsymbol{u}$,

$$\boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}(t) = e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

12: $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, $\det 9$, $\lambda = 3$ and 3 with *one* independent eigenvector $(1, 3)$.
14: When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ is $\|\boldsymbol{u}(0)\|$. So e^{At} is *orthogonal*.
17: (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. These show the unstable cases (a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$
21: $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{0} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e^t} & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$.
22: $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}\boldsymbol{At^2} + \frac{1}{6}At^3 + \cdots = I + (e^t - 1)A = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$.
26: (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \ne 0$. To see $e^{At}\boldsymbol{x}$, write $(I + At + \frac{1}{2}A^2t^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}\lambda^2 t^2 + \cdots)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$.