# 18.06 Solutions to PSet 6

**5.1:**

**3:** (a) *False*: $\det(I+I)$ is not $1+1$ (b) *True*: The product rule extends to $ABC$ (use it twice) (c) *False*: $\det(4A)$ is $4^n \det A$ (d) *False*: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible.

**15:** The first determinant is $0$, the second is $1 - 2t^2 + t^4 = (1-t^2)^2$.

**18:** $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$ (to reach 2 by 2, eliminate $a$ and $a^2$ in row 1 by column operations). Factor out $b-a$ and $c-a$ from the 2 by 2: $(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$.

**22:** $\det(A) = 3, \det(A^{-1}) = \frac{1}{3}, \det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. The numbers $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. *Note to instructor*: If you discuss this exercise, you can explain that this is the reason determinants come before eigenvalues. Identify $\lambda = 1$ and $\lambda = 3$ as the eigenvalues of $A$.

**5.2:**

**1:** $\det A = 1+18+12-9-4-6 = 12$, rows are independent; $\det B = 0$, row $1+$row $2 =$ row 3; $\det C = -1$, independent rows ($\det C$ has one term, odd permutation)

**5:** Four zeros in the same row guarantee $\det = 0$. $A = I$ has 12 zeros (maximum with $\det \neq 0$).

**8:** Some term $a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$ in the big formula is not zero! Move rows $1, 2, \ldots, n$ into rows $\alpha, \beta, \ldots, \omega$. Then these nonzero $a$'s will be on the main diagonal.

**24:** (a) All $L$'s have $\det = 1; \det U_k = \det A_k = 2, 6, -6$ for $k = 1, 2, 3$ (b) Pivots $2, \frac{3}{2}, \frac{-1}{3}$.

**25:** Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

**5.3:**

**4:** (a) $x_1 = \det\left(\left[\,\boldsymbol{b} \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3\,\right]\right) / \det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$ which is $x_1 \det A$.

**6:** (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

**10:** Take the determinant of $AC^{\mathrm{T}} = (\det A)I$. The left side gives $\det AC^{\mathrm{T}} = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $\det C = (\det A)^{n-1}$.

**23:** $A^{\mathrm{T}}A = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} & 0 & 0 \\ 0 & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} \end{bmatrix}$ has $\begin{array}{lcl} \det A^{\mathrm{T}}A & = & (\|a\|\|b\|\|c\|)^2 \\ \det A & = & \pm\|a\|\|b\|\|c\| \end{array}$

**39:** $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1} = 1/\det A$, construct $A^{-1}$ using the cofactors. *Invert to find* $A$.

**6.1:**

**3:** $A$ has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1,1)$ and $\boldsymbol{x}_2 = (2,-1)$. $A^{-1}$ has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and $-1$.

**9:** (a) Multiply by $A$: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$ (b) Multiply by $A^{-1}$: $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$ (c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A+I)\boldsymbol{x} = (\boldsymbol{\lambda}+\boldsymbol{1})\boldsymbol{x}$.

**12:** The projection matrix $P$ has $\lambda = 1, 0, 1$ with eigenvectors $(1,2,0)$, $(2,-1,0)$, $(0,0,1)$. Add the first and last vectors: $(1,2,1)$ also has $\lambda = 1$. *Note* $P^2 = P$ leads to $\lambda^2 = \lambda$ so $\lambda = 0$ or $1$.

**27:** $A$ has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of $A$). $C$ has rank 2 (ensuring two zero eigenvalues) and $(1,1,1,1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1,-1,1,-1)$.

**6.2:**

**1:** $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

**19:** $B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.