18.06 Solutions to PSet 5

4.2:

11: (a) $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (2,3,0)$, $\boldsymbol{e} = (0,0,4)$, $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$ (b) $\boldsymbol{p} = (4,4,6)$, $\boldsymbol{e} = \mathbf{0}$.

12: $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = projection matrix onto the column space of A (the xy plane)

$P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = Projection matrix onto the second column space. Certainly $(P_2)^2 = P_2$.

17: If $P^2 = P$ then $\boldsymbol{(I - P)^2} = (I-P)(I-P) = I - PI - IP + P^2 = \boldsymbol{I - P}$. When P projects onto the column space, $I - P$ projects onto the *left nullspace*.

26: A^{-1} exists since the rank is $r = m$. Multiply $A^2 = A$ by A^{-1} to get $A = I$.

32: Since $P_1\boldsymbol{b}$ is in $\boldsymbol{C}(A)$, $P_2(P_1\boldsymbol{b})$ equals $P_1\boldsymbol{b}$. So $P_2P_1 = P_1 = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ where $\boldsymbol{a} = (1,2,0)$.

4.3:

9: Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

12: (a) $\boldsymbol{a} = (1,\ldots,1)$ has $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = m$, $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = b_1 + \cdots + b_m$. Therefore $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/m$ is the **mean** of the b's (b) $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$ $\boldsymbol{b} = (1,2,b)$ $\|\boldsymbol{e}\|^2 = \sum_{i=1}^{m}(b_i - \widehat{x})^2 =$ **variance**

(c) $\boldsymbol{p} = (3,3,3)$ $\boldsymbol{e} = (-2,-1,3)$ $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{e} = 0$. $P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

17: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

28: Only 1 plane contains $\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless $\boldsymbol{a}_1, \boldsymbol{a}_2$ are *dependent*. Same test for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

4.4:

4: (a) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $QQ^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$. Any Q with $n < m$ has $QQ^{\mathrm{T}} \neq I$.

(b) $(1,0)$ and $(0,0)$ are *orthogonal*, not *independent*. Nonzero orthogonal vectors *are* independent. (c) Starting from $\boldsymbol{q}_1 = (1,1,1)/\sqrt{3}$ my favorite is $\boldsymbol{q}_2 = (1,-1,0)/\sqrt{2}$ and $\boldsymbol{q}_3 = (1,1,-2)/\sqrt{b}$.

12: (a) Orthonormal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1) = x_1$

(b) Orthogonal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1)$. Therefore $x_1 = \boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1$

(c) x_1 is the first component of A^{-1} times $\boldsymbol{b}$.

18: $\boldsymbol{A} = \boldsymbol{a} = (1, -1, 0, 0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. In $\mathbf{R}^5$, $\boldsymbol{D}$ would be $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$.

19: If $A = QR$ then $A^{\mathrm{T}}A = R^{\mathrm{T}}Q^{\mathrm{T}}QR = R^{\mathrm{T}}R =$ *lower* triangular times *upper* triangular (this Cholesky factorization of $A^{\mathrm{T}}A$ uses the same R as Gram-Schmidt!). The example has $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$ and the same R appears in $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = R^{\mathrm{T}}R$.

24: (a) One basis for the subspace $\boldsymbol{S}$ of solutions to $x_1 + x_2 + x_3 - x_4 = 0$ is $\boldsymbol{v}_1 = (1, -1, 0, 0)$, $\boldsymbol{v}_2 = (1, 0, -1, 0)$, $\boldsymbol{v}_3 = (1, 0, 0, 1)$ (b) Since $\boldsymbol{S}$ contains solutions to $(1, 1, 1, -1)^{\mathrm{T}}\boldsymbol{x} = 0$, a basis for $\boldsymbol{S}^{\perp}$ is $(1, 1, 1, -1)$ (c) Split $(1, 1, 1, 1) = \boldsymbol{b}_1 + \boldsymbol{b}_2$ by projection on $\boldsymbol{S}^{\perp}$ and $\boldsymbol{S}$: $\boldsymbol{b}_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ and $\boldsymbol{b}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$.