18.06 Solutions to PSet 4

3.5:

16: These bases are not unique! (a) $(1,1,1,1)$ for the space of all constant vectors (c,c,c,c) (b) $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components = 0 (c) $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $\boldsymbol{N}(I) = \{\text{zero vector}\}$.

26:

(a) $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$

(b) Add $\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$

(c) $\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}$.

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are $3, 6, 3$.

30: $\begin{bmatrix}-1&2&0\\0&0&0\end{bmatrix}, \begin{bmatrix}-1&0&2\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\-1&2&0\end{bmatrix}, \begin{bmatrix}0&0&0\\-1&0&2\end{bmatrix}$.

41: $\boldsymbol{I} = \begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix} - \begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix} + \begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix} + \begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix} - \begin{bmatrix}&&1\\1&&\\&1&\end{bmatrix}$. The six P's are dependent.

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 2nd cannot be (from $P_{32} = 1$) and also 5th ($P_{32} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

3.6:

6: A: $\dim$ $\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\boldsymbol{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: $\dim$ $\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\boldsymbol{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

14: Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

16: If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

32: The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

8.2:

8: $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ solving $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

9: Elimination on $A\boldsymbol{x} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

12.a: The nullspace and rank of $A^{\mathrm{T}}A$ and A are always the same.

4.1:

9: $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \mathbf{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$.

11: ***For A***: The nullspace is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The column space is the line through $(1, 3)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is the perpendicular line through $(3, -1)$.
For B: The nullspace of B is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$. The column space and left nullspace are the same as for A.

22: $(1, 1, 1, 1)$ is a basis for $\boldsymbol{P}^{\perp}$. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has $\boldsymbol{P}$ as its nullspace and $\boldsymbol{P}^{\perp}$ as row space

33: Both $\boldsymbol{r}$'s orthogonal to both $\boldsymbol{n}$'s, both $\boldsymbol{c}$'s orthogonal to both $\boldsymbol{\ell}$'s, each pair independent. All A's with these subspaces have the form $[\boldsymbol{c}_1\ \boldsymbol{c}_2]M[\boldsymbol{r}_1\ \boldsymbol{r}_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

4.2:

16: $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$. So $\boldsymbol{b}$ is in the plane. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.