18.06 Solutions to PSet 3

3.1.15 (a) Two planes through $(0,0,0)$ probably intersect in a line through $(0,0,0)$
(b) The plane and line probably intersect in the point $(0,0,0)$
(c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x}+\boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.
3.1.20 (a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + 2b_3$
3.1.24 The column space of AB is *contained in* (possibly equal to) the column space of A. The example $B = 0$ and $A \neq 0$ is a case when $AB = 0$ has a smaller column space than A.
3.2.18 Fill in **12** then **4** then **1** to get the complete solution to $x - 3y - z = 12$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{4} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{x}_{\text{particular}} + \boldsymbol{x}_{\text{nullspace}}$.
3.2.26 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice $\text{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
3.2.31 If $\boldsymbol{N}(A) =$ line through $\boldsymbol{x} = (2,1,0,1)$, A has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).
3.3.8 The new entries keep rank 1: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & \mathbf{4} & \mathbf{8} \\ 4 & \mathbf{8} & \mathbf{16} \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{2} & 6 & -\mathbf{3} \\ 1 & \mathbf{3} & -\mathbf{3/2} \\ 2 & 6 & -3 \end{bmatrix}$, $M = \begin{bmatrix} a & b \\ c & \boldsymbol{bc/a} \end{bmatrix}$.
3.3.19 We are given $AB = I$ which has rank n. Then $\text{rank}(AB) \leq \text{rank}(A)$ forces $\text{rank}(A) = n$. This means that A is invertible. The right-inverse B is also a left-inverse: $BA = I$ and $B = A^{-1}$.
3.3.22 $A = (\text{pivot columns})(\text{nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} \text{columns} \\ \text{times rows} \end{matrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$
3.4.6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$
(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

3.4.24 (a) $\begin{bmatrix}1\\1\end{bmatrix}[\,x\,] = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = [\,b\,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

3.4.28 $\begin{bmatrix}1 & 2 & 3 & \mathbf{0}\\0 & 0 & 4 & \mathbf{0}\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0 & \mathbf{0}\\0 & 0 & 1 & \mathbf{0}\end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix}-2\\1\\0\end{bmatrix}$; $\begin{bmatrix}1 & 2 & 3 & \mathbf{5}\\0 & 0 & 4 & \mathbf{8}\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0 & -\mathbf{1}\\0 & 0 & 1 & \mathbf{2}\end{bmatrix}$.
Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

3.4.34 (a) If $\boldsymbol{s} = (2, 3, 1, 0)$ is the only special solution to $A\boldsymbol{x} = \mathbf{0}$, the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ (line of solution!). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is *not free* in $\boldsymbol{s}$, and R must be $\begin{bmatrix}1 & 0 & -2 & 0\\0 & 1 & -3 & 0\\0 & 0 & 0 & 1\end{bmatrix}$.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ can be solve for all $\boldsymbol{b}$, because A and R have *full row rank* $r = 3$.

3.4.36 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b} =$ column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!