18.06 Solutions to PSet 2

2.3.1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2.3.17 The parabola $y = a + bx + cx^2$ goes through the 3 given points when $\begin{array}{l} a + b + c = 4 \\ a + 2b + 4c = 8 \\ a + 3b + 9c = 14 \end{array}$.
Then $a = 2$, $b = 1$, and $c = 1$. This matrix with columns $(1, 1, 1)$, $(1, 2, 3)$, $(1, 4, 9)$ is a "Vandermonde matrix."

2.4.6 $(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

2.4.32 A times $X = [\, \boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3 \,]$ will be the identity matrix $I = [\, A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3 \,]$.

2.4.34 $A*$ **ones** $= \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$ agrees with **ones** $*A = \begin{bmatrix} a+c & b+b \\ a+c & b+d \end{bmatrix}$ when $b = c$ and $a = d$
Then $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

2.5.10 $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block of B).

2.5.27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

2.5.29(a) True (If A has a row of zeros, then every AB has too, and $AB = I$ is impossible) (b) False (the matrix of all ones is singular even with diagonal 1's: *ones* (3) has 3 equal rows) (c) True (the inverse of A^{-1} is A and the inverse of A^2 is $(A^{-1})^2$).

2.5.42
$$\begin{aligned} MM^{-1} &= (I_n - UV)\,(I_n + U(I_m - VU)^{-1}V) \quad \text{(this is testing formula } \mathbf{3}\text{)} \\ &= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V \quad \text{(keep simplifying)} \\ &= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n \quad \text{(formulas } \mathbf{1}, \mathbf{2}, \mathbf{4} \text{ are similar)} \end{aligned}$$

2.6.5 $EA = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$. With E^{-1} as L, $A = LU = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix} U$.

2.6.17 (a) L goes to I (b) I goes to L^{-1} (c) LU goes to U. Elimination multiply by L^{-1}!

2.6.24 The upper left blocks all factor at the same time as A: A_k is L_kU_k.

2.7.2 $(AB)^{\mathrm{T}}$ is not $A^{\mathrm{T}}B^{\mathrm{T}}$ *except when* $AB = BA$. Transpose that to find: $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$.

2.7.20 $\begin{bmatrix}1 & 3\\ 3 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -7\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$; $\begin{bmatrix}1 & b\\ b & c\end{bmatrix} = \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & c-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}$

$$\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix} = \begin{bmatrix}1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1\end{bmatrix}\begin{bmatrix}2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3}\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2} & 0\\ & 1 & -\frac{2}{3}\\ & & 1\end{bmatrix} = \boldsymbol{LDL}^{\mathrm{T}}.$$

2.7.38 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$P = \begin{bmatrix}P_2 & \\ & P_3\end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $P_3 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$ and $P^6 = I$.

2.7.40 Start from $Q^{\mathrm{T}}Q = I$, as in $\begin{bmatrix}\boldsymbol{q}_1^{\mathrm{T}}\\ \boldsymbol{q}_2^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(a) The diagonal entries give $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 = 1$: *unit vectors*

(b) The off-diagonal entry is $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ (and in general $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$)

(c) The leading example for Q is the rotation matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.