18.06 Solutions to PSet 1

1.1.12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.
1.1.26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. *The solution is* $c = 2$ *and* $d = 4$. Then $2(1, 2) + 4(3, 1) = (14, 8)$.
1.2.27 The length $\|\boldsymbol{v} - \boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

1.3.3
$$\begin{aligned} y_1 &= B_1 \\ y_1 + y_2 &= B_2 \\ y_1 + y_2 + y_3 &= B_3 \end{aligned} \quad \text{gives} \quad \begin{aligned} y_1 &= B_1 \\ y_2 &= -B_1 + B_2 \\ y_3 &= -B_2 + B_3 \end{aligned} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: independent columns in A and S!

1.3.4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$.

1.3.6 $c = 3 \quad \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & \mathbf{3} \end{bmatrix}$ has column $3 = 2$ (column 1) $+$ column 2

$c = -1 \quad \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has column $3 = -$ column $1 +$ column 2

$c = 0 \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ has column $3 = 3$ (column 1) $-$ column 2

2.1.22 The dot product $A\boldsymbol{x} = \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (1 by 3)(3 by 1) is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.
2.1.32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*
2.1.35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices.
6 row orders $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)
2.2.8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.
2.2.32 The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A =$ **eye**(100). A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through** $\mathbf{0}$. The column picture has 100 vectors all in the same 99-dimensional hyperplane.