18.06 Quiz 1 Professor Strang February 28, 2011

Your PRINTED name is ____________________ **1.**

Your Recitation Instructor (and time) is ____________________ **2.**

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1. (a) By elimination find the **rank** of A and the pivot columns of A (in its column space):

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}.$$

(b) Find the special solutions to $Ax = 0$ and then find **all solutions** to $Ax = 0$.

(c) For which number b_3 does $Ax = \begin{bmatrix} 3 \\ 9 \\ b_3 \end{bmatrix}$ have a solution?

Write the **complete solution** x (the general solution) with that value of b_3.

(a) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$r = \text{rank}(A) = 2$, pivot columns are $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}$

(b) Special solutions: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. $N(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

all solutions $\qquad c_1, c_2 \in \mathbb{R}$

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 3 & 6 & 3 & 9 & 9 \\ 2 & 4 & 2 & 9 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & b_3-6 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & b_3-6 \end{array}\right]$

Hence to have a solution we need $b_3 - 6 = 0 \Rightarrow \boxed{b_3 = 6}$

For this value of b_3, a particular solution is given by $x_p = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

complete solution: $x_c = x_p + x_n = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

nullsolution

2. Suppose A is a 3 by 5 matrix and the equation $Ax = b$ has a solution for every b. What are $(a)(b)(c)(d)$? (If you don't have enough information to answer, tell as much about the answer as you can.)

(a) Column space of A : Since $AX = b \Rightarrow b = x_1 A_1 + \cdots + x_n A_n$ where $A_1, \cdots A_n$ are the Columns of A, every b in $\mathbb{R}^3$ is in $C(A)$. So $C(A) = \mathbb{R}^3$.

(b) Nullspace of A Since $C(A) = \mathbb{R}^3$, we have $r = 3$ and therefore # free variables = # special Solutions $= n - r = 5 - 3 = 2$. Hence $N(A)$ is a plane in $\mathbb{R}^5$.

(c) Rank of A

By (a) $C(A) = \mathbb{R}^3$ therefore $r = 3$. We can also argue by saying that $r = m$ or we have a constraint on b.

(d) Rank of the 6 by 5 matrix $B = \begin{bmatrix} A \\ A \end{bmatrix}$.

We can use elimination to obtain

$$\begin{bmatrix} A \\ A \end{bmatrix} \longrightarrow \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

But $\begin{bmatrix} A \\ 0 \end{bmatrix}$ has rank 3. Therefore $\text{rank}(B) = 3$.

3. (a) When an odd permutation matrix P_1 multiplies an even permutation matrix P_2, the product $P_1 P_2$ is ___odd___ (EXPLAIN WHY).

P_1 applies an odd number of row exchanges to I and P_2 applies an even number. Hence $P_1 P_2$ applies odd + even = odd number of row exchanges.

(b) If the columns of B are vectors in the nullspace of A, then AB is ___0 matrix___ (EXPLAIN WHY). Let $B = [B_1 | \cdots | B_K]$, where $B_1, \ldots, B_K$ are the columns of B. Since each B_i is in $N(A)$ we have $AB_i = 0$.

Then $AB = [AB_1 | \cdots\cdots | AB_K] = [0 | \cdots | 0] = 0$

(c) If $c = 0$, factor this matrix into $A = LU$ (lower triangular times upper triangular):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}.$$

(d) That matrix A is invertible unless $c =$ ___21___.

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 6 & c-3 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & c-21 \end{bmatrix}$

Hence A is invertible unless $c - 21 = 0 \Rightarrow \boxed{c = 21}$

(a) when $c = 0$ we get $A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$