18.06 Problem Set 9

Due Wednesday, 29 April 2009 at 4pm in 2-106. Total = 130 points.

1. (30 points = 5+5+5+5+5+5) Let $A = \begin{pmatrix} 1 & s \\ 1 & 3 \end{pmatrix}$, where s is some real number.

   (a) Give a value of s where A is defective; use this s in the subsequent parts.

   (b) Compute a set of eigenvectors and generalized eigenvectors (as defined in the handout) of A to give a complete basis for $\mathbb{R}^2$. (Use this basis in the subsequent parts.)

   (c) For the column vector $\mathbf{u}_0 = (1,0)$, compute $\mathbf{u}(t) = (I + e^{At})^{-1}\mathbf{u}_0$ (as an explicit formula with no matrix operations). (Hint: use the formula for $f(A)$ from the handout; note that $I = A^0$.)

   (d) For the column vector $\mathbf{u}_0 = (1,0)$, compute $\mathbf{u}_k = A^k\mathbf{u}_0$ (as an explicit formula with no matrix operations).

   (e) Write an explicit formula for A^k, for any k (as an explicit formula with no matrix operations). (Consider your answer for the previous part, and ask what matrix you would multiply by an arbitrary vector to obtain A^k times that vector.)

   (f) Suppose we perturb the matrix slightly, changing s to $s + 0.0001$. Does $\|\mathbf{u}_k\|$ grow more slowly or more quickly with k than when A was defective?

2. (25 points = 5+5+5+5+5) True or false, with a good reason:

   (a) A can't be similar to $-A$ unless $A = 0$.

   (b) An invertible matrix can't be similar to a singular matrix.

   (c) A symmetric matrix can't be similar to a nonsymmetric matrix.

   (d) Any diagonalizable matrix is similar to a Hermitian matrix.

   (e) If B is invertible, then AB and BA have the same eigenvalues.

3. (15 points = 5+5+5) This question concerns the second-order ODE $y'' + 10y' + 25y = 0$ with the initial conditions $y(0) = 2$, $y'(0) = 3$.

   (a) Convert this into a matrix equation $d\mathbf{u}/dt = A\mathbf{u}$ by $u_1 = y$, $u_2 = y'$. The initial condition is $\mathbf{u}(0) =$ __________.

   (b) Find the eigenvalues and eigenvectors of A. A is a __________ matrix.

   (c) Find the solution $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$, and hence the solution $y(t)$.

4. (10 points) Suppose that λ_1 is a double root of $\det(A - \lambda_1 I)$ for some A, but that $N(A - \lambda_1 I)$ is one dimensional, the span of a single eigenvector $\mathbf{x}_1$. A is thus defective. It turns out that $(A - \lambda_1 I)^2$ must always have a *two*-dimensional nullspace if λ_1 is a double root. Let $\mathbf{y}$ be a vector in the nullspace of $(A - \lambda_1 I)^2$ that is *not* in $N(A - \lambda_1 I)$. In what nullspace is $(A - \lambda_1 I)\mathbf{y}$? Hence, $(A - \lambda_1 I)\mathbf{y}$ is proportional to __________.

5. (15 points) This is a Matlab problem using the SVD to perform image compression. This is not the best technique for image compression, but it showcases the SVD's ability to extract the important information from a matrix.

   (a) Download `http://jdj.mit.edu/~stevenj/strang.jpg`, a grayscale image of a familiar fellow, to the directory that you launch Matlab from (e.g. your home directory on Athena). Each pixel is stored as a number from 0 (black) to 255 (white), so the image can be interpreted as a matrix A (in this case, a 404×303 matrix). Read it into Matlab and display it with the commands:

```
A = flipud(double(imread('strang.jpg')));
pcolor(A); shading interp; colormap('gray'); axis equal
```

(b) Now, compute the SVD $A = U\Sigma V^T$ using Matlab, create a new figure, and plot the distribution of singular values σ_i (the diagonals of Σ) on a log scale:

```
[U,S,V] = svd(A);
figure
semilogy(diag(S), 'o');
xlabel('index of singular value'); ylabel('singular values');
```

(c) Now, let's see what happens if we throw out all but the biggest 50 singular values, just setting the other ones to zero to make a new matrix S2:

```
S2 = S * diag([ones(1,50), zeros(1,size(S,2)-50)]);
figure
pcolor(U*S2*V'); shading interp; colormap('gray'); axis equal
```

It should still look a lot like the original image: most of the information is in the biggest singular values and the corresponding singular vectors!

(d) Replace the two 50's in the previous commands to a smaller number, to keep fewer than 50 singular values. How small can you go before the image becomes unrecognizable? Which details of the image are the last to be blurred away?

6. (15 points) Find the eigenvalues and orthonormal eigenvectors of A^TA and AA^T for the Fibonacci matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Construct the singular value decomposition and verify that $A = U\Sigma V^T$.

7. (10 points) If $A = QR$ with an orthogonal matrix Q (A is square), the SVD of A is almost the same as the SVD of R. Which of the three matrices U, Σ, and V must be different for A and R?

8. (10 points) Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are two orthonormal bases for $\mathbb{R}^n$. Write a formula for the matrix A that transforms each $\mathbf{v}_j$ into $\mathbf{u}_j$ to give $A\mathbf{v}_1 = \mathbf{u}_1$, ..., $A\mathbf{v}_n = \mathbf{u}_n$. A is a/an ____________________ matrix (hint: check A^TA).