18.06 Problem Set 4

Due Wednesday, 11 March 2008 at 4pm in 2-106.

1. A is an $m \times n$ matrix of rank r. Suppose there are right-hand-sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

   (a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n, and r?

   (b) $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$. Why must this be true?

2. A is an $m \times n$ matrix of rank r. Which of the four fundamental subspaces are the same for:

   (a) A and $\begin{pmatrix} A \\ A \end{pmatrix}$

   (b) $\begin{pmatrix} A \\ A \end{pmatrix}$ and $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$

   Explain why all three matrices A, $\begin{pmatrix} A \\ A \end{pmatrix}$, and $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ must have the same rank r.

3. Find a basis for each of the four subspaces for

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

4. True or false (give a reason if true, or a counterexample if false):

   (a) A and A^T have the same number of pivots

   (b) A and A^T have the same left nullspace

   (c) If the $C(A) = C(A^T)$, then $A = A^T$.

   (d) If $A^T = -A$, then the row space of A is the same as the column space of A.

5. Use the Matlab command `A = rand(10,5);` to make a random 10×5 matrix A, and `B = rand(5,9)` to make a random 5×9 matrix B. Then use the command `[R,p] = rref(A*B);` to find the row-reduced echelon form R and a list p of the pivot columns. Using this information, give bases for the nullspace, column space, and row space of AB.

6. Explain why the following statement must be true: if a subspace S is contained in another subspace V, then the orthogonal complement $V^\perp$ is contained in the orthogonal complement $S^\perp$.

7. If $A^TA\mathbf{x} = 0$ then $A\mathbf{x} = 0$. Reason: $A^TA\mathbf{x} = 0$ means that $A\mathbf{x}$ in the nullspace of ____________. $A\mathbf{x}$ is also in the ____________ space of A. These two spaces are ____________, so their only intersection is $A\mathbf{x} = 0$. Thus, A^TA has the same nullspace as A. (We derive this in another way in class.)

8. Suppose you have two matrices V and W such that $C(V)$ and $C(W)$ are orthogonal subspaces. What is V^TW?

9. Suppose L is a one-dimensional subspace (a line through the origin) in $\mathbb{R}^3$. Its orthogonal complement $L^\perp$ is the ____________ perpendicular to L. Then $(L^\perp)^\perp$ is a ____________ perpendicular to $L^\perp$, and in fact $(L^\perp)^\perp$ is the same as ____________.

10. Let N be a matrix whose columns are a basis for the nullspace of A. Then the nullspace of N^T is the ____________ space of A.

11. Let A be the matrix

$$A = \begin{pmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{pmatrix}.$$

   (a) Find the projection matrix P_C onto $C(A)$.

   (b) Find the projection matrix P_R onto the row space of A

   (c) Compute $P_C A P_R$. Explain your result.

   (d) For *any* matrix A (not necessarily the one above), with P_C and P_R defined as the projection matrices onto A's column and row space respectively, conclude that you would get $P_C A P_R$ = ____________.

12. Find the projection matrix P onto the plane $x + 2y - z = 0$ in two ways:

   (a) Choose two vectors in the plane and make them the columns of a matrix A. The plane is the column space. Then compute $P = A(A^T A)^{-1} A^T$.

   (b) Write a vector $\mathbf{e}$ that is perpendicular to that plane. Compute the matrix $Q = \mathbf{e}\mathbf{e}^T / \mathbf{e}^T \mathbf{e}$ that projects onto the $\mathbf{e}$ direction. Then compute $\mathbf{P} = I - Q$.

13. The nullspace of A^T is ____________ to the column space $C(A)$, so if $A^T \mathbf{b} = 0$ then the projection of $\mathbf{b}$ onto $C(A)$ should be $\mathbf{p} =$ ____________. Check that $P\mathbf{b}$ gives this answer, where P is the projection matrix $P = A(A^T A)^{-1} A^T$.

14. Explain why one must have $P^2 = P$, from the definition of the projection matrix P onto the column space of a matrix A (if we take a vector $\mathbf{b}$ and project it to the column space to get $P\mathbf{b}$, then project it again, we must get ____________). Check explicitly that this is true from the formula $P = A(A^T A)^{-1} A^T$.