18.06 Problem Set 4 Solution

Due Wednesday, 11 March 2009 at 4 pm in 2-106.
Total: 175 points.

Problem 1: A is an $m \times n$ matrix of rank r. Suppose there are right-hand-sides $\vec{b}$ for which $A\vec{x} = \vec{b}$ has no solution.

(a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) $A^{\mathrm{T}}\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$. Why must this be true?

Solution (15 points = 10+5)

(a) First of all, the rank r of a matrix is the number of column (row) pivots, it must be less than equal to m and n. If the matrix were of full row rank, i.e., $r = m$, it would imply that $A\vec{x} = \vec{b}$ always has a solution; we know that this is not the case, and hence $r \neq m$. To sum up, the inequalities among m, n, r are $r \leq n, r < m$.

(b) Since A^{T} is an $n \times m$ matrix, the null space $N(A^{\mathrm{T}})$ has dimension $m - r$, which is positive by (a). Hence, $A^{\mathrm{T}}\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$.

Problem 2: A is an $m \times n$ matrix of rank r. Which of the four fundamental subspaces are the same for:

(a) A and $\begin{pmatrix} A \\ A \end{pmatrix}$

(b) $\begin{pmatrix} A \\ A \end{pmatrix}$ and $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$

Explain why all three matrices A, $\begin{pmatrix} A \\ A \end{pmatrix}$, and $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ must have the same rank r.

Solution (20 points = 10+10)

(a) Note that if we do invertible row operations on the matrix $\begin{pmatrix} A \\ A \end{pmatrix}$, we may kill the bottom A and get $\begin{pmatrix} A \\ 0 \end{pmatrix}$.

Nullspace: Since the nullspace is invariant under row operations, the two matrices have the same nullspace.

Column space: Since the two matrices do not have the same number of rows, their column space must not be the same.

Row space: Since the row space is also invariant under row operations, the two matrices have the same row space.

Left nullspace: The transpose of $\begin{pmatrix} A \\ A \end{pmatrix}$ is $\begin{pmatrix} A^{\mathrm{T}} & A^{\mathrm{T}} \end{pmatrix}$. Hence, the left nullspace is a subspace of $\mathbb{R}^{2m}$. However, the left nullspace is a subspace of $\mathbb{R}^m$. They cannot be the same.

(b) Using invertible column operations, we can turn $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ into $\begin{pmatrix} A & 0 \\ A & 0 \end{pmatrix}$.

The nullspace and the row space of $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ are subspaces of $\mathbb{R}^{2n}$, whereas the nullspace and the row space of $\begin{pmatrix} A \\ A \end{pmatrix}$ are subspaces of $\mathbb{R}^n$. They cannot be the same.

Since the column space and the left nullspace are invariant under column operations, the two matrices have the same column space and left nullspace.

REMARK: For an $m \times n$ matrix of rank r, we have

Fundamental space	subspace of	dimension
Nullspace	$\mathbb{R}^n$	$n - r$
Column space	$\mathbb{R}^m$	r
Row space	$\mathbb{R}^n$	r
Left nullspace	$\mathbb{R}^m$	$m - r$

Problem 3: Find a basis for each of the four subspaces for

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Solution (25 points = 5(echelon form)+5+5+5+5)

We first write A as in row-reduced echelon form.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The second and the fourth variables are the pivots. The first, third, and the fifth variables are free variables.

The row operation matrix E ($R = EA$) is

$$E = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Nullspace: The null space is as follow:

$$N(A) = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \qquad \text{for } x_1, x_3, x_5 \in \mathbb{R}.$$

Column space: Since the pivot columns (the second and the fourth) of A span the column space, we have

$$C(A) = a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \qquad \text{for } a, b \in \mathbb{R}$$

CAUTION: We get the pivot columns by looking at R but $C(A)$ is spanned by these columns of A, NOT these columns of R.

Row space: it is the same as the row space of R as R is obtained by invertible row operations. So

$$C(A^{\mathrm{T}}) = x_2 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \qquad \text{for } x_2, x_4 \in \mathbb{R}.$$

Left nullspace: It has a basis given by the rows of E for which the corresponding rows of R are all zero. That is to say, we need to take the last row of E. Thus,

$$N(A^{\mathrm{T}}) = a \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \text{for } a \in \mathbb{R}.$$

Problem 4: True or false (give a reason if true, or a counterexample if false):

(a) A and A^{T} have the same number of pivots

(b) A and A^{T} have the same left nullspace

(c) If the $C(A) = C(A^{\mathrm{T}})$, then $A = A^{\mathrm{T}}$.

(d) If $A^{\mathrm{T}} = -A$, then the row space of A is the same as the column space of A.

Solution (20 points = 5+5+5+5)

(a) True, because A and A^{T} have the same rank, which equals to the number of pivots of the matrices.

(b) False. In particular, if A is an $m \times n$ matrix of rank r with $m \neq n$, the dimension of two left nullspace will not be the same; Indeed, $\dim N(A^{\mathrm{T}}) = m - r$ and $\dim N\big((A^{\mathrm{T}})^{\mathrm{T}}\big) = n - r$.

For example, $A = (1\ 0)$. The left nullspace of A is $N(A^{\mathrm{T}}) = 0$ and the left null space of A^{T} is $N(A) = a \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $a \in \mathbb{R}$.

(c) This is not necessarily true. In particular, if A is a non-symmetric $n \times n$ invertible matrix, $C(A) = C(A^{\mathrm{T}}) = \mathbb{R}^n$.

(d) True, because the row space of A is $C(A^{\mathrm{T}}) = C(-A) = C(A)$ is the same as the column space of A.

Problem 5: Use the Matlab command `A = rand(10,5);` to make a random 10×5 matrix A, and `B = rand(5,9)` to make a random 5×9 matrix B. Then use the command `[R,p] = rref(A*B);` to find the row-reduced echelon form R and a list p of the pivot columns. Using this information, give bases for the nullspace, column space, and row space of AB.

Solution (10 points)

```
>> A = rand(10, 5)

A =

   0.8147    0.1576    0.6557    0.7060    0.4387
   0.9058    0.9706    0.0357    0.0318    0.3816
   0.1270    0.9572    0.8491    0.2769    0.7655
```

```
   0.9134    0.4854    0.9340    0.0462    0.7952
   0.6324    0.8003    0.6787    0.0971    0.1869
   0.0975    0.1419    0.7577    0.8235    0.4898
   0.2785    0.4218    0.7431    0.6948    0.4456
   0.5469    0.9157    0.3922    0.3171    0.6463
   0.9575    0.7922    0.6555    0.9502    0.7094
   0.9649    0.9595    0.1712    0.0344    0.7547

>> B = rand(5, 9)

B =

   0.2760    0.4984    0.7513    0.9593    0.8407    0.3500    0.3517    0.2858    0.0759
   0.6797    0.9597    0.2551    0.5472    0.2543    0.1966    0.8308    0.7572    0.0540
   0.6551    0.3404    0.5060    0.1386    0.8143    0.2511    0.5853    0.7537    0.5308
   0.1626    0.5853    0.6991    0.1493    0.2435    0.6160    0.5497    0.3804    0.7792
   0.1190    0.2238    0.8909    0.2575    0.9293    0.4733    0.9172    0.5678    0.9340

>> A*B

ans =

   0.9286    1.2919    1.8685    1.1771    1.8386    1.1234    1.5918    1.3642    1.3783
   0.9837    1.4991    1.3083    1.5080    1.3997    0.7170    1.5133    1.2495    0.5212
   1.3780    1.6044    1.6448    1.0018    1.8204    0.9787    2.1912    1.9411    1.4428
   1.2960    1.4439    2.0233    1.4829    2.4020    1.0544    2.0258    1.8017    1.3699
   1.2012    1.4129    1.2570    1.2013    1.4851    0.6973    1.5093    1.4414    0.7016
   0.8119    1.0343    1.5049    0.5253    1.3908    0.9914    1.4975    1.2978    1.5163
   1.0164    1.3029    1.5755    0.8194    1.5298    1.0059    1.6739    1.4764    1.3959
   1.1588    1.6152    1.6404    1.2939    1.6898    0.9712    1.9498    1.6330    1.1498
   1.4711    2.1755    2.5493    1.7674    2.4308    1.5765    2.5515    2.1319    1.8662
   1.1261    1.6491    1.7527    1.6739    1.9043    0.9477    1.9478    1.5730    0.9475

>> [R,p] = rref(A*B)

R =

   1.0000         0         0         0         0    0.2954   -2.6341   -0.5533   -0.7131
        0    1.0000         0         0         0   -0.1327    3.4223    1.5519    0.9401
        0         0    1.0000         0         0    1.1299   -2.1805   -0.9454    0.5034
        0         0         0    1.0000         0   -0.0369   -1.6062   -0.8889   -1.2387
```

```
         0         0         0         0    1.0000   -0.5696    3.0357    1.4608    0.7307
         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0         0


p =

     1     2     3     4     5
```

The pivot variables are $x_1, \ldots, x_5$ and free variables $x_6, \ldots, x_9$.

The column space is spanned by the first 5 columns of AB. The row space of AB is the same as the row space of R which is generated by the first five rows of R. The nullspace is given by the negative of the upper right 5×4 block together with a 4×4 identity matrix, one on top of the other. In other words, it is the column space of the following matrix.

$$\begin{pmatrix} -0.2954 & 2.6341 & 0.5533 & 0.7131 \\ 0.1327 & -3.4223 & -1.5519 & -0.9401 \\ -1.1299 & 2.1805 & 0.9454 & -0.5034 \\ 0.0369 & 1.6062 & 0.8889 & 1.2387 \\ 0.5696 & -3.0357 & -1.4608 & -0.7307 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 6: Explain why the following statement must be true: if a subspace S is contained in another subspace V, then the orthogonal complement $V^{\perp}$ is contained in the orthogonal complement $S^{\perp}$.

Solution (5 points)

Let $x \in V^{\perp}$ be an element in the orthogonal complement. We want to show that it is in the orthogonal complement $S^{\perp}$; that is to say for any $s \in S$, $x^{\mathrm{T}}s = 0$. But we know that $x^{\mathrm{T}}v = 0$ for any $v \in V$ and in particular, s is an element in a subspace of V. So, $x^{\mathrm{T}}s = 0$. We are done.

REMARK: Since the orthogonal complement of a subspace S is defined to be the subspace whose vectors pairs to zero with the vectors in S, the intuition here is that the larger the S is, the more restriction $S^\perp$ has, and hence the smaller $S^\perp$ is.

Problem 7: If $A^T A\vec{x} = 0$ then $A\vec{x} = 0$. Reason: $A^{\mathrm{T}} A\vec{x} = 0$ means that $A\vec{x}$ in the nullspace of __________. $A\vec{x}$ is also in the __________ space of A. These two spaces are __________, so their only intersection is $A\vec{x} = 0$. Thus, $A^{\mathrm{T}}A$ has the same nullspace as A. (We derive this in another way in class.)

Solution (5 points)

A^{T}; column space; orthogonal.

Problem 8: Suppose you have two matrices V and W such that $C(V)$ and $C(W)$ are orthogonal subspaces. What is $V^{\mathrm{T}}W$?

Solution (5 points)

It has to be zero. Indeed, if we write $V = (\vec{v}_1 \ \cdots \ \vec{v}_n)$ and $W = (\vec{w}_1 \cdots \vec{w}_m)$, $v_i \in C(V)$ and $w_j \in C(W)$ and

$$V^{\mathrm{T}}W = \begin{pmatrix} \vec{v}_1^{\mathrm{T}}\vec{w}_1 & \cdots & \vec{v}_1^{\mathrm{T}}\vec{w}_m \\ \vdots & \ddots & \vdots \\ \vec{v}_n^{\mathrm{T}}\vec{w}_1 & \cdots & \vec{v}_n^{\mathrm{T}}\vec{w}_m \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

because $C(V)$ and $C(W)$ are orthogonal.

Problem 9: Suppose L is a one-dimensional subspace (a line through the origin) in $\mathbb{R}^3$. Its orthogonal complement $L^\perp$ is the __________ perpendicular to L. Then $(L^\perp)^\perp$ is a __________ perpendicular to $L^\perp$, and in fact $(L^\perp)^\perp$ is the same as __________.

Solution (5 points)

plane; this is because $\dim L^\perp + \dim L = 3$.
line; this is because $\dim L^\perp + \dim(L^\perp)^\perp = 3$.
L.

It is true in general that for a subspace V of $\mathbb{R}^n$, $(V^\perp)^\perp = V$.

Problem 10: Let N be a matrix whose columns are a basis for the nullspace of A. Then the nullspace of N^{T} is the __________ space of A.

Solution (5 points)

row.

The condition implies that the nullspace of A is the same as the column space of N, i.e.$N(A) = C(N)$. We also know that the null space of N^{T} (or the left nullspace of N) is the orthogonal complement of the column space of N, i.e. $N(N^{\mathrm{T}}) = C(N)^{\perp}$, and the row space of A is the orthogonal complement of the nullspace of A, i.e. $C(A^{\mathrm{T}}) = N(A)^{\perp}$. Hence, $N(N^{\mathrm{T}}) = C(N)^{\perp} = N(A)^{\perp} = C(A^{\mathrm{T}})$.

REMARK1: The condition that the columns of N form a basis is not necessary, we need only that the column space of N is the same as the nullspace of A.

REMARK2: For a matrix A, the column space is orthogonal to the left nullspace and the row space is orthogonal to the nullspace.

Problem 11: Let A be the matrix

$$A = \begin{pmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{pmatrix}.$$

(a) Find the projection matrix P_C onto $C(A)$.

(b) Find the projection matrix P_R onto the row space of A

(c) Compute $P_C A P_R$. Explain your result.

(d) For *any* matrix A (not necessarily the one above), with P_C and P_R defined as the projection matrices onto A's column and row space respectively, conclude that you would get $P_C A P_R =$ ____________.

Solution (20 points = 5+5+5+5)

(a) The column space $C(A)$ is spanned by the vector $a = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. So the projection P_C onto $C(A)$ is obtained via dividing aa^{T} by $a^{\mathrm{T}}a$, which is

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} \Big/ (3 \times 3 + 4 \times 4) = \begin{pmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{pmatrix}.$$

(b) The row space $C(A^{\mathrm{T}})$ is spanned by the vector $b = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$. So the projection

P_R onto $C(A^{\mathrm{T}})$ is

$$bb^{\mathrm{T}}/b^{\mathrm{T}}b = \begin{pmatrix}3\\6\\6\end{pmatrix}\begin{pmatrix}3 & 6 & 6\end{pmatrix}\Big/(3\times 3+6\times 6+6\times 6) = \begin{pmatrix}1/9 & 2/9 & 2/9\\2/9 & 4/9 & 4/9\\2/9 & 4/9 & 4/9\end{pmatrix}.$$

(c)

$$\begin{aligned}P_C A P_R &= \begin{pmatrix}9/25 & 12/25\\12/25 & 16/25\end{pmatrix}\begin{pmatrix}3 & 6 & 6\\4 & 8 & 8\end{pmatrix}\begin{pmatrix}1/9 & 2/9 & 2/9\\2/9 & 4/9 & 4/9\\2/9 & 4/9 & 4/9\end{pmatrix}\\ &= \begin{pmatrix}3 & 6 & 6\\4 & 8 & 8\end{pmatrix}\begin{pmatrix}1/9 & 2/9 & 2/9\\2/9 & 4/9 & 4/9\\2/9 & 4/9 & 4/9\end{pmatrix} = \begin{pmatrix}3 & 6 & 6\\4 & 8 & 8\end{pmatrix} = A\end{aligned}$$

See the explanation below.

(d) A.

This consists of two parts: $P_C A = A$ and $AP_R = A$.

First, multiplying the projection matrix P_C on the left is equivalent to projecting each column of A onto the subspace $C(A)$. This will just give back A itself. Thus, $P_C A = A$.

Similarly, multiplying the projection matrix P_R on the right is equivalent to projecting each row of A onto the subspace $C(A^{\mathrm{T}})$. Thus, we will get back to A.

An alternative way to see this is to check it for each vector x. We can write any vector x as $x = P_R x + z$, where z is in the complement of the row space, which means that x is in the null space. Therefore $Ax = A(P_R x + z) = AP_R x + Az = AP_R x$ since $Az = 0$, and hence $A = AP_R$ since this is true for all x.

Problem 12: Find the projection matrix P onto the plane $x + 2y - z = 0$ in two ways:

(a) Choose two vectors in the plane and make them the columns of a matrix A. The plane is the column space. Then compute $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

(b) Write a vector $\vec{e}$ that is perpendicular to that plane. Compute the matrix $Q = \vec{e}\vec{e}^{\mathrm{T}}/\vec{e}^{\mathrm{T}}\vec{e}$ that projects onto the $\vec{e}$ direction. Then compute $\vec{P} = I - Q$.

Solution (20 points = 10+10)

(a) We view the plane as the null space of the matrix $(1\ 2\ -1)$. Then a basis of this space is given by $\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$. Hence we take the matrix A to be

$$A = \begin{pmatrix}-2 & 1\\ 1 & 0\\ 0 & 1\end{pmatrix}.$$

Thus, $A^{\mathrm{T}}A = \begin{pmatrix}-2 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}-2 & 1\\ 1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}5 & -2\\ -2 & 2\end{pmatrix}$. and

$$(A^{\mathrm{T}}A)^{-1} = \begin{pmatrix}1/3 & 1/3\\ 1/3 & 5/6\end{pmatrix}.$$

Then the projection matrix is

$$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = \begin{pmatrix}-2 & 1\\ 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1/3 & 1/3\\ 1/3 & 5/6\end{pmatrix}\begin{pmatrix}-2 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} = \begin{pmatrix}5/6 & -1/3 & 1/6\\ -1/3 & 1/3 & 1/3\\ 1/6 & 1/3 & 5/6\end{pmatrix}.$$

(b) Since the row space of the matrix $(1\ 2\ -1)$ is orthogonal to the nullspace, which is the plane as explained in part (a), we may take $e = \begin{pmatrix}1\\2\\-1\end{pmatrix}$. Thus, we have

$$Q = ee^{\mathrm{T}}/e^{\mathrm{T}}e = \begin{pmatrix}1 & 2 & -1\\ 2 & 4 & -2\\ -1 & -2 & 1\end{pmatrix}\Big/ 6 = \begin{pmatrix}1/6 & 1/3 & -1/6\\ 1/3 & 2/3 & -1/3\\ -1/6 & -1/3 & 1/6\end{pmatrix}.$$

Hence,

$$P = I - Q = \begin{pmatrix}5/6 & -1/3 & 1/6\\ -1/3 & 1/3 & 1/3\\ 1/6 & 1/3 & 5/6\end{pmatrix}.$$

Problem 13: The nullspace of A^{T} is ____________ to the column space $C(A)$, so if $A^{\mathrm{T}}\vec{b} = 0$ then the projection of $\vec{b}$ onto $C(A)$ should be $\vec{p} =$ ____________. Check that $P\vec{b}$ gives this answer, where P is the projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Solution (10 points)

orthogonal.

0; the condition says that $\vec{b}$ is in the orthogonal complement of the column space $C(A)$ and hence it has projection 0 on that space.

We calculation $P\vec{b}$ as follows.

$$P\vec{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b = A(A^{\mathrm{T}}A)^{-1}\vec{0} = 0$$

Problem 14: Explain why one must have $P^2 = P$, from the definition of the projection matrix P onto the column space of a matrix A (if we take a vector $\vec{b}$ and project it to the column space to get $P\vec{b}$, then project it again, we must get ____________). Check explicitly that this is true from the formula $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Solution (10 points)

$P\vec{b}$; this is because we can write P^2b as $P(P\vec{b})$ and the projection restricted to the column space of A is identity, so further composition of $P\vec{b}$ with P will still give $P\vec{b}$.

If $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$, we have

$$\begin{aligned}
P\vec{b} &= A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\vec{b} \\
P\vec{b} &= \left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2\vec{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\vec{b} \\
&= A\Big((A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)\Big)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\vec{b} = A \cdot I \cdot (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\vec{b} = P\vec{b}.
\end{aligned}$$