18.06 Problem Set 3

Due Wednesday, 25 February 2008 at 4pm in 2-106.

1. Consider the matrix
$$A = \begin{pmatrix} 1 & 2 & 1 & 4 & 1 \\ 2 & 6 & 3 & 11 & 1 \\ 1 & 4 & 2 & 7 & 0 \end{pmatrix}$$
   (a) Reduce A to echelon form U, find a special solution for each free variable, and hence describe all solutions to $Ax = 0$.
   (b) By further row operations on U, find the reduced echelon form R.
   (c) True or false: $N(R) = N(U)$?
   (d) True or false: $C(A) = C(U)$?

2. If you do column elimination steps (instead of row eliminations) on a matrix A to get some other matrix U (like in problem 6 of pset 1), does $N(A) = N(U)$? Come up with a counter-example if false, or give an explanation why this should always hold if true.

3. Suppose that column 3 of a 4×6 matrix is all zero. Then x_3 must be a __________ variable. Give one special solution for this matrix.

4. Fill in the missing numbers to make the matrix A rank 1, rank 2, and rank 3. (i.e. your solution should be three matrices).
$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}.$$

5. Suppose A and B have the *same* reduced echelon form R. Therefore A equals a/an ________________ matrix multiplying B on the __________ (left or right).

6. Write the complete solution (i.e. a particular solution plus all nullspace vectors) to the system:
$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

7. Explain why these statements are *all false* by giving a counter-example for each:
   (a) A system $Ax = b$ has at most one particular solution.
   (b) A system $Ax = b$ has at least one particular solution.
   (c) If there is only one special solution x_n in the nullspace and there exists some particular solution x_p, then the complete solution to $Ax = b$ is any linear combination of x_p and x_n.
   (d) If A is invertible then there is no solution x_n the nullspace.
   (e) The solution x_p with all free variables set to zero is the "shortest" solution (minimizing $\|x\|$).

8. If A is a 3×7 matrix, its largest possible rank is __________. In this case, there is a pivot in every __________ of U and R, the solution to $Ax = b$ __________ (*always exists* or *is unique*), and the column space of A is __________. Construct an example of such a matrix A.

9. If A is a 6×3 matrix, its largest possible rank is __________. In this case, there is a pivot in every __________ of U and R, the solution to $Ax = b$ __________ (*always exists* or *is unique*), and the nullspace of A is __________. Construct an example of such a matrix A.

10. Find the rank of A, $A^{T}A$, and AA^{T}, for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$.

11. Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

12. Find a basis for the space of 2×3 matrices whose nullspace contains $(1,2,0)$.

13. Make the matrix $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ in Matlab by the command:
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>> A = [2 1; 6 3]
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Then compute $b = Ax$ for 100 random x vectors by the command:
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>> br = A * rand(2, 100);
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Plot these b vectors as black dots by the commands:
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>> plot(br(1,:), br(2,:), 'k.')
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What is the pattern, and why?