18.06 Professor Strang Quiz 3 May 2, 2008

Grading

1

2

3

Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	afr
2)	M 2	4-149	A. Tievsky	2-492	3-4093	tievsky
3)	M 3	2-131	A. Ritter	2-085	2-1192	afr
4)	M 3	2-132	A. Tievsky	2-492	3-4093	tievsky
5)	T 11	2-132	J. Yin	2-333	3-7826	jbyin
6)	T 11	8-205	A. Pires	2-251	3-7566	arita
7)	T 12	2-132	J. Yin	2-333	3-7826	jbyin
8)	T 12	8-205	A. Pires	2-251	3-7566	arita
9)	T 12	26-142	P. Buchak	2-093	3-1198	pmb
10)	T 1	2-132	B. Lehmann	2-089	3-1195	lehmann
11)	T 1	26-142	P. Buchak	2-093	3-1198	pmb
12)	T 1	26-168	P. McNamara	2-314	4-1459	petermc
13)	T 2	2-132	B. Lehmann	2-089	2-1195	lehmann
14)	T 2	26-168	P. McNamara	2-314	4-1459	petermc

1 (40 pts.) The (real) matrix A is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & x & 3 \\ 2 & 3 & 6 \end{bmatrix}.$$

(a) What can you tell me about the eigenvectors of A?

What is the sum of its eigenvalues?

(b) For which values of x is this matrix A positive definite?

(c) For which values of x is A^2 positive definite? **Why**?

(d) If R is any **rectangular** matrix, *prove* from $x^{\mathrm{T}}(R^{\mathrm{T}}R)x$ that $R^{\mathrm{T}}R$ is positive semidefinite (or definite). What condition on R is the test for $R^{\mathrm{T}}R$ to be positive definite?

Solution (10+10+10+10 points)

a) Since A is a symmetric matrix (no matter what x is), its eigenvectors may be chosen orthonormal (5 points). The sum of the eigenvalues is the same as the trace of A, that is, the sum of the diagonal entries: $tr(A) = 7 + x$.

b) In this case, the easiest tests for positive definiteness are the pivot test and the determinant test. I'll use the determinant test.

A matrix A is positive definite when every one of the top-left determinants is positive (3 points for correct defn.). In this case, the three determinants are 1, $x - 1$, and

$$\det(A) = 1(6x - 9) - (6 - 6) + 2(3 - 2x) = 2x - 3. \quad (1)$$

(6 points). All of these are positive precisely when $x > 3/2$ (1 point).

c) Perhaps the clearest way to think about this is by using the eigenvalues. Suppose A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$. (They are all real because A is symmetric.) Then the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \lambda_3^2$ (5 points). These are all positive so long as the eigenvalues are non-zero. So, A^2 is positive definite except when A has an eigenvalue of 0, or equivalently, except when A is not invertible (3 points). We found in part that $\det(A) = 0$ only when $x = 3/2$. Thus, the final answer is that A^2 is positive definite except when $x = 3/2$ (2 points).

One could also find A^2 explicitly and use the determinant or pivot test. In practice this turned out to lead to a lot of mistakes. However, you could notice that the top left entry of A^2 is 6, the 2 by 2 determinant is $6(10 + x^2) - (7 + x)^2 = 5x^2 - 14x + 11 > 0$, and the 3 by 3 determinant is $\det(A^2) = \det(A)^2$. The only way that any of these could be non-positive is if $\det(A) = 0$.

A final approach is to follow the steps for part d) below.

d) We use the $x^T Ax$ test for positive (semi)definiteness. We have

$$x^T R^T Rx = (Rx)^T Rx = Rx \cdot Rx \tag{2}$$

This is just the length of the vector Rx. This length is positive when Rx is not the zero vector and is 0 when Rx is the 0 vector. In particular, since this number is always at least 0, $R^T R$ is definitely positive semidefinite (6 points). It is positive definite when this number is positive for any nonzero x. That is, we need for Rx to only be the 0 vector when x is the 0 vector. This is equivalent to saying that R has trivial nullspace, or R has full column rank (4 points).

2 (30 pts.) The **cosine** of a matrix is defined by copying the series for $\cos x$ (which always converges):

$$\cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) Suppose $Ax = \lambda x$. Show that x is an eigenvector of $\cos A$. Find the eigenvalue.

(b) Find the eigenvalues of $A = \frac{\pi}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvectors are $(1,1)$ and $(1,-1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $\cos A$.

(c) The second derivative of the series for $\cos(At)$ is $-A^2\cos(At)$. So $u(t) = \mathbf{\cos(At)u(0)}$ is a short formula for the solution of

$$\frac{d^2u}{dt^2} = -A^2 u \quad \text{starting from} \quad u(0) \quad \text{with} \quad u'(0) = 0\,.$$

Now construct that $u(t) = \cos(At)u(0)$ by the usual three steps when A is diagonalizable: $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, $Ax_3 = \lambda_3 x_3$.

1. Expand $u(0) = c_1x_1 + c_2x_2 + c_3x_3$ in the eigenvectors.
2. Multiply those eigenvectors by ____, ____, ____.
3. Add up the solution $u(t) = c_1$ ____ $x_1 + c_2$ ____ $x_2 + c_3$ ____ x_3.

Solution (10+10+10 points)

a) Suppose that $Ax = \lambda x$. Then

$$\begin{aligned}
\cos(A)x &= Ix - \frac{1}{2!}A^2x + \frac{1}{4!}A^4x - \ldots && (3)\\
&= x - \frac{1}{2!}\lambda^2 x + \frac{1}{4!}\lambda^4 x - \ldots && (4)\\
&= \left(1 - \frac{1}{2!}\lambda^2 + \frac{1}{4!}\lambda^4 - \ldots\right)x && (5)\\
&= \cos(\lambda)x && (6)
\end{aligned}$$

So x is an eigenvector of $\cos(A)$ with eigenvalue $\cos(\lambda)$.

b) We define

$$A = \frac{\pi}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{7}$$

We know that $(1,1)$ and $(1,-1)$ are eigenvectors of A. We can find the eigenvalues simply by acting by A:

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\pi}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \pi\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8}$$

So A has eigenvalue $\lambda_1 = \pi$. Similarly,

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

So A has eigenvalue $\lambda_2 = 0$ (4 points). Just as for any other function $(A^2, e^A, A^{-1}, \ldots)$, this means that $\cos(A)$ has eigenvectors $(1,1)$ with eigenvalue $\cos(\pi) = -1$ and $(1,-1)$ with eigenvalue $\cos(0) = 1$ (3 points). We can put these into the diagonalization formula to find $\cos(A)$:

$$\cos(A) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{10}$$

(3 points)

c) This problem is modeled after what happens for e^{At}. After expanding $u(0)$, step 2 involves multiplying the eigenvectors by $\cos(\lambda_1 t)$, $\cos(\lambda_2 t)$, and $\cos(\lambda_3 t)$. So the final answer is

$$u(t) = c_1\cos(\lambda_1 t)x_1 + c_2\cos(\lambda_2 t)x_2 + c_3\cos(\lambda_3 t)x_3 \tag{11}$$

(10 points) Some common mistakes were forgetting to include the t, using the function e instead of cos, or putting in something entirely different for the coefficients.

3 (30 pts.) Suppose the vectors x, y give an orthonormal basis for $\mathbf{R}^2$ and $A = xy^{\mathrm{T}}$.

(a) Compute the rank of A and the rank of $A^2 = (xy^{\mathrm{T}})(xy^{\mathrm{T}})$. Use this information to find the eigenvalues of A.

(b) Explain why this matrix B is **similar** to A (*and write down what similar means*):

$$B = \begin{bmatrix} x^{\mathrm{T}} \\ y^{\mathrm{T}} \end{bmatrix} A \begin{bmatrix} x & y \end{bmatrix}$$

(c) The eigenvalues of Q are $\lambda_1 = e^{i\theta} = \cos\theta + i\sin\theta$ and $\lambda_2 = e^{-i\theta} = \cos\theta - i\sin\theta$:

$$\text{Rotation matrix } \; Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Find the eigenvectors of Q. Are they perpendicular ?

a) Any matrix given by $A = xy^T$ for two non-zero vectors x,y will have rank 1. Every row will be a multiple of y^T, and every column will be a multiple of x, meaning that it must have rank 1. Alternatively, we note that x is in the nullspace of A, and that y is not, so that A must have rank exactly 1. (3 points)

Note that $A^2 = (xy^T)(xy^T) = x(y^Tx)y^T$ is the zero matrix, since $y^Tx = 0$ (they are perpendicular vectors). So A^2 has rank 0. (3 points)

If the eigenvalues of A are λ_1 and λ_2, then the eigenvalues of A^2 are λ_1^2 and λ_2^2. Since A^2 only has the eigenvalue 0, both λ_1 and λ_2 must be 0. (4 points)

b) Two square matrices A and B are similar if there is some invertible matrix M such that $B = MAM^{-1}$ (5 points). Similarity is *not* the same thing as having equal eigenvalues; this only works if both A and B are diagonalizable matrices, and in fact our A is not diagonalizable. To be more precise, similarity implies that A and B have equal eigenvalues, but the converse is not true.

In this case we check that A and B are similar by showing that the other factors are inverses.

$$\begin{bmatrix} x^{\mathrm{T}} \\ y^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x^T x & x^T y \\ y^T x & y^T y \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

The last step is true because x and y are perpendicular, and both of unit length (5 points).

c) Given the eigenvalues of Q, we find the eigenvectors using $N(Q - \lambda I)$. We start with $\lambda_1 = \cos\theta + i\sin\theta$:

$$Q - (\cos\theta + i\sin\theta)I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \tag{14}$$

and this matrix has nullspace generated by $(1, -i)$ or equivalently $(i, 1)$. Similarly, for $\lambda_2 = \cos\theta - i\sin\theta$ we find

$$Q - (\cos\theta - i\sin\theta)I = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} \tag{15}$$

which has nullspace generated by $(1, i)$. (8 points)

Every orthogonal matrix has perpendicular eigenvectors. We check in this specific case:

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}^H \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{17}$$

$$= 0 \tag{18}$$

(2 points) Make sure to take $x_1^H x_2$ and not $x_1^T x_2$.