18.06 Professor Strang Quiz 2 April 2, 2008

Grading

1

2

3

Your PRINTED name is: SOLUTIONS

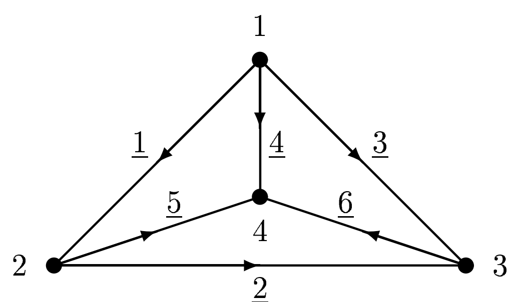
Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	afr
2)	M 2	4-149	A. Tievsky	2-492	3-4093	tievsky
3)	M 3	2-131	A. Ritter	2-085	2-1192	afr
4)	M 3	2-132	A. Tievsky	2-492	3-4093	tievsky
5)	T 11	2-132	J. Yin	2-333	3-7826	jbyin
6)	T 11	8-205	A. Pires	2-251	3-7566	arita
7)	T 12	2-132	J. Yin	2-333	3-7826	jbyin
8)	T 12	8-205	A. Pires	2-251	3-7566	arita
9)	T 12	26-142	P. Buchak	2-093	3-1198	pmb
10)	T 1	2-132	B. Lehmann	2-089	3-1195	lehmann
11)	T 1	26-142	P. Buchak	2-093	3-1198	pmb
12)	T 1	26-168	P. McNamara	2-314	4-1459	petermc
13)	T 2	2-132	B. Lehmann	2-089	2-1195	lehmann
14)	T 2	26-168	P. McNamara	2-314	4-1459	petermc

1 (33 pts.) (a) If $Ax = b$ and $A^{\mathrm{T}}y = 0$ then b is perpendicular to y. (The column space of A is perpendicular to the nullspace of A^{T}.) **Prove this by computing $(Ax)^{\mathrm{T}}y$.**

(b) Write down the 6 by 4 incidence matrix A of this graph (1 and -1 in each row of A). What is the dimension of the column space $C(A)$? Describe the nullpace $N(A)$.

(c) Find one nonzero vector $y = (y_1, y_2, \ldots, y_6)$ that is in the nullspace of A^{T}. (Think loops.) If voltages x_1, x_2, x_3, x_4 are assigned to the nodes (keep the x's as variables not numbers), multiply by A to find Ax. **Check that this Ax is perpendicular to your vector y.** (That's Kirchhoff's Voltage Law.)



Solution (6+13+14 points)

a) We have

$$(Ax)^T y = x^T A^T y = x^T (0) = 0 \tag{1}$$

b) To form the incidence matrix A, for each edge we put a -1 for the node where the edge starts, and a 1 for the node where the edge ends. (5 points)

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{2}$$

Any incidence matrix has a one-dimensional nullspace spanned by the vector consisting of all 1. Thus, $C(A)$ has dimension $r = n - \dim(N(A)) = 4 - 1 = 3$, and the nullspace has the vector $(1, 1, 1, 1)$ as a basis. (8 points) Of course this can also be calculated by hand.

c) To find the left nullspace of an incidence matrix, we traverse around a closed loop, and keep track of the edges with signs. So for example if we go from node 1 to node 2 to node 4 and back to node 1, we find the vector $(1, 0, 0, -1, 1, 0)$ in the left nullspace of A. (7 points) Some other examples are: $(1, 1, -1, 0, 0, 0)$, $(0, 0, 1, -1, 0, 1)$, and $(0, 1, 0, 0, -1, 1)$.

The matrix Ax is (3 points)

$$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix} \tag{3}$$

Taking the dot product of Ax with the first choice of y above yields (4 points)

$$(Ax)^T y = (x_2 - x_1) - (x_4 - x_1) + (x_4 - x_2) = 0 \tag{4}$$

2 (33 pts.) (a) Suppose you want to fit the best straight line $C + Dt$ to the values $b = 1, 1, 1, 2$ at the times $t = 0, 1, 3, 4$. What is the matrix A in the unsolvable system $A \begin{bmatrix} C \\ D \end{bmatrix} = b$? Find the best $\widehat{C}, \widehat{D}$ and the heights p_1, p_2, p_3, p_4 of that line $\widehat{C} + \widehat{D}t$ at the times $t = 0, 1, 3, 4$.

(b) Think of the same problem as a projection onto the column space of A in $\mathbf{R}^4$. What is the error vector $e = b - p$? Show with numbers that e is perpendicular to (what space?).

(c) Use Gram-Schmidt to get orthonormal columns q_1, q_2 from the columns a_1, a_2 of your matrix A.

Solution (11+11+11 points)

a) The matrix A is (4 points)

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \tag{5}$$

To find the best $\widehat{C}, \widehat{D}$, we need to solve the system (2 points)

$$A^T A \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \tag{6}$$

or

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \end{bmatrix} \tag{8}$$

We obtain the solution $(\widehat{C}, \widehat{D}) = (17/20, 1/5)$. (2 points)

To find the height vector p, we take (3 points)

$$p = A \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{20}[17, 21, 29, 33]^T \tag{9}$$

b) The error vector $e = [1, 1, 1, 2]^T - p = \frac{1}{20}[3, -1, -9, 7]^T$. (3 points) The error vector is perpendicular to the column space of A. (4 points) We check using numbers (4 points):

$$[1, 1, 1, 1]e = \frac{1}{20}(3 - 1 - 9 + 7) = 0 \tag{10}$$

$$[0, 1, 3, 4]e = \frac{1}{20}(0 - 1 - 27 + 28) = 0 \tag{11}$$

c) We use Gram-Schmidt on the columns of A. We set $w_1 = a_1$ (3 points), and then (5 points)

$$w_2 = a_2 - \frac{w_1 \cdot a_2}{w_1 \cdot w_1} w_1 = [-2, -1, 1, 2]^T \tag{12}$$

Finally, we must normalize to obtain $q_1 = \frac{1}{2}[1, 1, 1, 1]^T$ and $q_2 = \frac{1}{\sqrt{10}}[-2, -1, 1, 2]^T$. (3 points)

3 (34 pts.) This question is about the matrix

$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

(a) Compute A^2 and use that to show that the determinant of A is either 1 or -1.

(b) Determine whether $\det A = 1$ or -1.

(c) Find the cofactor C_{11} corresponding to the entry $a_{11} = -\frac{1}{2}$.

(d) Out of the $4! = 24$ terms in the "big formula" for $\det A$, show **four terms** that are $+\frac{1}{16}$. (For each term give the column numbers like $4, 3, 2, 1$ or $2, 1, 4, 3$ as you go down the matrix.)

Solution (9+9+8+8 points)

a) We have (4 points)

$$A^2 = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I \tag{13}$$

Thus, $\det(A)^2 = \det(I) = 1$, meaning that $\det(A) = \pm 1$. (5 points) In fact A is a symmetric orthogonal matrix, which means that we know many of its properties.

b) Now we actually compute the determinant of A. One could do this using cofactors, row reduction, etc. It turns out to be -1. (9 points for computation) For example, after row-reducing the first column of A we obtain

$$B = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix} \tag{14}$$

Then

$$\begin{aligned} \det(A) &= \det(B) && (15) \\ &= \frac{1}{16}\det\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix} && (16) \\ &= \frac{1}{16}(-1)\det\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} && (17) \\ &= -\frac{1}{2}(1+1) = -1 && (18) \end{aligned}$$

c) Let D be the matrix we get when we cross out row 1 and column 1 of A. Then C_{11} is the determinant of D (there is no negative sign since $(1,1)$ is a "positive" position: $(-1)^{1+1} = 1$). (4 points) The calculation yields $\frac{1}{2}$. (4 points)
A quicker way to do it is to use the inverse formula. We have found that $A = A^{-1}$. We also know that the quantity $C_{11}/\det(A)$ is equal to the top left entry of A^{-1}. Thus, we must have $C_{11} = a_{11}\det(A) = \frac{1}{2}$.

d) Each term of the big formula is a product, where we take one entry from each row and each column. To find the terms that are $+\frac{1}{16}$, we need to have the signs cancel out correctly. Using the notation of the problem, the choices are $(1,2,3,4)$, $(2,1,4,3)$, $(3,4,1,2)$, and $(4,3,2,1)$. (2 points each)