18.06 Professor Strang Quiz 1 March 3, 2008

Your PRINTED name is: ____________________

Grading

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Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	afr
2)	M 2	4-149	A. Tievsky	2-492	3-4093	tievsky
3)	M 3	2-131	A. Ritter	2-085	2-1192	afr
4)	M 3	2-132	A. Tievsky	2-492	3-4093	tievsky
5)	T 11	2-132	J. Yin	2-333	3-7826	jbyin
6)	T 11	8-205	A. Pires	2-251	3-7566	arita
7)	T 12	2-132	J. Yin	2-333	3-7826	jbyin
8)	T 12	8-205	A. Pires	2-251	3-7566	arita
9)	T 12	26-142	P. Buchak	2-093	3-1198	pmb
10)	T 1	2-132	B. Lehmann	2-089	3-1195	lehmann
11)	T 1	26-142	P. Buchak	2-093	3-1198	pmb
12)	T 1	26-168	P. McNamara	2-314	4-1459	petermc
13)	T 2	2-132	B. Lehmann	2-089	2-1195	lehmann
14)	T 2	26-168	P. McNamara	2-314	4-1459	petermc

1 (18 pts.) Start with an invertible 3 by 3 matrix A. Construct B by subtracting 4 times row 1 of A from row 3. **How do you find B^{-1} from A^{-1}?** You can answer in matrix notation, but *you must also answer in words*—what happens to the columns and rows?

Solution (18 points)

We can find B by multiplying A on the left by an appropriate matrix E_{31} (remember, row operations correspond to multiplication on the left, column operations correspond to multiplication on the right). Here, we need

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \tag{1}$$

Since $B = E_{31}A$, we know $B^{-1} = (E_{31}A)^{-1} = A^{-1}E_{31}^{-1}$. Thus, B^{-1} can be found by doing some column operations on A^{-1}.

What operations specifically? We know

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \tag{2}$$

(this is the standard pattern for an E matrix). This represents adding 4 times column 3 to column 1.

2 (24 pts.) Elimination on A leads to U:

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{leads to} \quad Ux = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Factor the first matrix A into $A = LU$ and also into $A = LDL^{\mathrm{T}}$.

(b) Find the inverse of A by Gauss-Jordan elimination on $AA^{-1} = I$ or by inverting L and D and L^{T}.

(c) If D is diagonal, show that LDL^{T} is a symmetric matrix for every matrix L (square or rectangular).

Solution (9+9+6 points)

a) Since A is symmetric, we have an LDL^T decomposition, and we find this first. Since

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \tag{3}$$

we must have

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{4}$$

and so

$$L^T = D^{-1}U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Of course, this means that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{6}$$

which we could have calculated directly.

b) If we did it using Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 3 & 7 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 6 & -1 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 4 & 0 & -1 & 1 \end{array}\right]$$

(This is L^{-1} on the right hand side, so we can multiply and solve. We may also continue.)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 4 & 0 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & 1/4 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1/4 & -1/4 \\ 0 & 1 & 0 & -1/2 & 3/4 & -1/4 \\ 0 & 0 & 1 & 0 & -1/4 & 1/4 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 3/4 & -1/4 \\ 0 & 0 & 1 & 0 & -1/4 & 1/4 \end{array}\right]$$

c) We show a matrix is symmetric by showing that it is equal to its transpose. If D is diagonal, then of course $D = D^T$. Thus:

$$\begin{aligned} (LDL^T)^T &= (L^T)^T D^T L^T \\ &= LDL^T \end{aligned}$$

3 (30 pts.) Suppose the nonzero vectors a_1, a_2, a_3 point in different directions in R^3 but

$$3a_1 + 2a_2 + a_3 = \text{zero vector}\,.$$

The matrix A has those vectors a_1, a_2, a_3 in its columns.

(a) Describe the nullspace of A (all x with $Ax = 0$).

(b) Which are the pivot columns of A?

(c) I want to show that *all* 3 by 3 matrices with

$$(*) \qquad 3(\text{column 1}) + 2(\text{column 2}) + (\text{column 3}) = \text{zero vector}$$

form a **subspace** S of the space M of 3 by 3 matrices. Now the zero matrix is certainly included.

Suppose B and C are matrices whose columns have this property **(*)**. To show that we have a subspace, we have to prove that every linear combination of B and C ___(finish sentence)___ .

Go ahead and prove that.

Solution (10+10+10 points)

a) First, the problem gives us one vector in the nullspace: the equation

$$3a_1 + 2a_2 + a_3 = \text{zero vector} \tag{7}$$

is the same as saying that $Ax = 0$, where

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{8}$$

Thus, the nullspace $N(A)$ contains all scalar multiples of this vector $(3, 2, 1)$. Technically, we need to argue that there is nothing else in the nullspace. You must recognize that the matrix has rank 2 either here or in part b to get full credit.

b) Since we have a linear relationship between the columns, column 3 can not be a pivot column, so the rank is at most 2. The problem also tell us that the three a_i vectors point in different directions. This means that they can't all be on the same line; that is, the column space must be at least a plane. Thus the the rank is exactly 2.
Having rank 2 means that there are 2 pivot columns and one free column. Since column 3 is a linear combination of columns before it, it must be a free column. So columns 1 and 2 are pivots, and column 3 is free.

c) To show that we have a subspace, we have to prove that every linear combination of B and C <u>also has property (*)</u> .
Suppose that B and C have property (*). Consider the matrix $D = t_1 B + t_2 C$ for any two numbers t_1, t_2. For ease of notation, I'll denote the ith column of a matrix A by $\text{col}_i(A)$. We have

$$\text{col}_i(D) = t_1 \text{col}_i(B) + t_2 \text{col}_i(C) \tag{9}$$

We can now check that D also satisfies (*):

$$\begin{aligned}
3\text{col}_1(D) + 2\text{col}_2(D) + \text{col}_3(D) &= 3\left(t_1 \text{col}_1(B) + t_2 \text{col}_1(C)\right) \\
&\quad +2\left(t_1 \text{col}_2(B) + t_2 \text{col}_2(C)\right) \\
&\quad +\left(t_1 \text{col}_3(B) + t_2 \text{col}_3(C)\right) \\
&= t_1\left(3\text{col}_1(B) + 2\text{col}_2(B) + \text{col}_3(B)\right) \\
&\quad +t_2\left(3\text{col}_1(C) + 2\text{col}_2(C) + \text{col}_3(C)\right) \\
&= 0
\end{aligned}$$

4 (28 pts.) Start with this 2 by 4 matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix}$$

(a) Find all special solutions to $Ax = 0$ and **describe the nullspace** of A.

(b) Find the complete solution—meaning all solutions (x_1, x_2, x_3, x_4)—to

$$Ax = \begin{bmatrix} 2x_1 + 3x_2 + x_3 - x_4 \\ 6x_1 + 9x_2 + 3x_3 - 2x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(c) When an m by n matrix A has rank $r = m$, the system $Ax = b$ can be solved for which b (best answer)? How many special solutions to $Ax = 0$?

Solution (10+8+10 points)

a) We find the special solutions by reducing A. I'll go all the way to row reduced echelon form:

$$\begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, in turn we set each free variable to 1 and the rest to 0, and solve $Ux = 0$. The special solutions are

$$s_1 = \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$s_2 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

The nullspace is all linear combinations of these two vectors; it will be a plane in $\mathbb{R}^4$.

b) One way to solve for a particular solution is just to look at the set-up: our b is the negative of the 4th column, so $(0, 0, 0, -1)$ will work.

We could also solve for a particular solution using an augmented matrix:

$$\begin{bmatrix} 2 & 3 & 1 & -1 & 1 \\ 6 & 9 & 3 & -2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

We can now back substitute and solve. We can pick the free variables to be whatever we like; we may as well set them to be 0. Then the bottom equation gives us $x_4 = -1$, and the top then gives $x_1 = 0$. This is the same vector as above.

The complete solution is particular solution plus the nullspace:

$$x_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_1 \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{12}$$

c) When an m by n matrix has rank m, the dimension of the column space (= rank) is the same as the dimension of the ambient space. Thus, every vector is in the column space; the equation $Ax = b$ can be solved for every b. The number of special solutions will be the number of rows minus the rank (note that $n \geq m$ since the rank is no greater than n). Thus, we find that there are $n - m$ special solutions.