18.06 Problem Set 8

Due Wednesday, 23 April 2008 at 4 pm in 2-106.

Problem 1: Do problem 3 in section 6.5 (pg. 339) in the book.

Problem 2: Do problem 6 in section 6.5 (pg. 339).

Problem 3: For what numbers c and d are the matrices A and B positive definite? Test the 3 determinants:

$$A = \begin{bmatrix} c & 2 & 3 \\ 2 & c & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & d & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

Problem 4: Do problem 15 in section 6.5 (pg. 340).

Problem 5: Do problem 28 in section 6.5 (pg. 342).

Problem 6: a) Let $f_1(x, y) = \frac{1}{4}x^4 + x^2y + y^2$. Find the second derivative matrix

$$A_1 = \begin{bmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 \end{bmatrix}$$

A_1 is not positive definite everywhere - find the conditions on x and y for it to be positive definite. (Interesting question: check what happens when both partial derivatives vanish, that is, when $\partial f/\partial x = \partial f/\partial y = 0$. Is f still positive definite? It turns out that f_1 does attain a minimal value, but not along isolated points. Find the points where it hits a global minimum. This part is not required.)

b) Let $f_2(x, y) = x^3 + xy - x$. Find the second derivative matrix A_2. When is this matrix positive definite? (Interesting question: check what happens when both partial derivatives vanish, so when $\partial f/\partial x = \partial f/\partial y = 0$. Show that you get a saddle point. This part is not required.)

Problem 7: Do problem 3 in section 8.1 (pg. 410).

Problem 8: Do problem 11 in section 8.1 (pg. 411). This problem requires Matlab.

Here, we are thinking of u as a vector whose components represent a potential solution at the points 0, 1/8, 2/8, ... that satisfies the boundary conditions $u(0) = u(1) = 0$. So u has 9 components, where the first and last components are 0 and the rest are variables that we are solving for.

In this case, discretizing the differential equation amounts to solving a matrix equation $Mu = f$, where M is a matrix found by combining the differential operators, and f is the constant vector $[1, 1, 1, \ldots]$. The operator du/dx can be approximated by multiplying u by the difference matrix

$$A = \frac{1}{1/8}\begin{bmatrix} 1 & 0 & 0 & \ldots \\ -1 & 1 & 0 & \ldots \\ 0 & -1 & 1 & \ldots \\ \vdots & \vdots & \ddots & \end{bmatrix}$$

Here, 1/8 represents the change Δx, and the matrix takes successive differences $u(x_0) - u(x_0 - \Delta x)$, so the resulting vector has entries approximating the slopes of u. The matrix A should have 9 columns (one for each entry of u) and 9 rows (one for each difference, plus an extra first row).

We can also define another difference matrix A^T; this represents $-du/dx$ because now we are taking differences $u(x_0) - u(x_0 + \Delta x)$. This will again be 9 by 9. Finally, $-d^2u/dx^2$ can be approximated by multiplying by the second difference matrix

$$A^TA = \frac{1}{1/64}\begin{bmatrix} 2 & -1 & 0 & 0 & \ldots \\ -1 & 2 & -1 & 0 & \ldots \\ 0 & -1 & 2 & -1 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

So, the question asks us to solve the two systems $(A^TA + 10A)u = [1, 1, 1, \ldots]^T$ (using the forward difference) and $(A^TA - 10A^T)u = [1, 1, 1, \ldots]^T$ (using the backwards difference), subject to the conditions $u(0) = u(1) = 0$. The easiest way to model the boundary conditions is simply to cut off the top and bottom entries of u, and then to cut off the first and last columns and rows of the matrix multiplying it. This will leave a vector u with 7 components and a multiplier that is also 7 by 7. Then we can use Matlab to solve $Mu = [1, 1, 1, \ldots]^T$.

To plot points, first create a vector x containing the x coordinates. Then

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plot(x,u,'+k')
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will show a graph of your points marked by + symbols. To differentiate your two results, the command

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plot(x,u,'xk')
```

will use x symbols instead.