18.06 Problem Set 8

Due Wednesday, 23 April 2008 at 4 pm in 2-106.

Problem 1: Do problem 3 in section 6.5 (pg. 339) in the book.

Solution (10 points)

The matrix A encodes the quadratic $f = x^2 + 4xy + 9y^2$. Completing the square is the same as finding the row reduced form of A: we find

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

so $f = (x + 2y)^2 + 5y^2$.

Similarly, B gives the quadratic $f = x^2 + 6xy + 9y^2$. This time the reduced form is

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

so $f = (x + 3y)^2$.

Problem 2: Do problem 6 in section 6.5 (pg. 339).

Solution (10 points)

If A has full column rank, we know that A^TA is square symmetric and invertible. We must show that it is also positive definite. The easiest criterion is to show that for every non-zero vector x the number $x^T(A^TA)x > 0$. To do this we just note that $x^TA^TAx = Ax \cdot Ax = \|Ax\|^2 > 0$.

Problem 3: For what numbers c and d are the matrices A and B positive definite? Test the 3 determinants:

$$A = \begin{bmatrix} c & 2 & 3 \\ 2 & c & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & d & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

Solution (15 points)

The three top-left determinants of A are c, $c^2 - 4$, and $c(c-16) - 2(-10) + 3(8-3c) = c^2 - 25c + 44$. If A is positive definite, all three of these numbers must be positive. First of all $c > 0$. The second determinant yields

$$c^2 - 4 = (c-2)(c+2) > 0$$

so either $c > 2$ or $c < -2$. Because $c > 0$ we can ignore the second case. Finally, the third determinant $c^2 - 25c + 44$ has roots $25/2 \pm \frac{1}{2}\sqrt{449}$ and we either need c to be smaller than the smaller root or larger than the larger root. The smaller root is less than 2, so we can ignore that piece of it. In the end we find

$$c > \frac{25 + \sqrt{449}}{2}$$

The top three determinants of B are 1, $d-4$, and $(d-9)-2(-1)+1(6-d) = -1$. This last determinant is negative no matter what d is, so this matrix will never be positive definite.

Problem 4: Do problem 15 in section 6.5 (pg. 340).

Solution (10 points)

We must show that if A and B are positive definite, so is $A+B$. We use the $x^T(A+B)x$ criterion: we have $x^T(A+B)x = x^TAx + x^TBx$. If x is nonzero than both terms are positive (since A and B are positive definite), so the whole thing is positive. Thus $A+B$ is also positive definite.

Problem 5: Do problem 28 in section 6.5 (pg. 342).

Solution (10 points)

The key is that this is a $Q\Lambda Q^T$ decomposition of A.

a) The determinant of A is the product of the eigenvalues, so $\det A = 10$.

b) The eigenvalues of A are 2 and 5.

c) the eigenvectors of A are $[\cos\theta, \sin\theta]^T$ and $[-\sin\theta, \cos\theta]^T$.

d) A is symmetric since it has a $Q\Lambda Q^T$ decomposition. It is then positive definite because the eigenvalues are positive.

Problem 6: a) Let $f_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$. Find the second derivative matrix

$$A_1 = \begin{bmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 \end{bmatrix}$$

A_1 is not positive definite everywhere - find the conditions on x and y for it to be positive definite. (Interesting question: check what happens when both partial derivatives vanish, that is, when $\partial f/\partial x = \partial f/\partial y = 0$. Is f still positive definite? It turns out that f_1 does attain a minimal value, but not along isolated points. Find the points where it hits a global minimum. This part is not required.)

b) Let $f_2(x, y) = x^3 + xy - x$. Find the second derivative matrix A_2. When is this matrix positive definite? (Interesting question: check what happens when both partial derivatives vanish, so when $\partial f/\partial x = \partial f/\partial y = 0$. Show that you get a saddle point. This part is not required.)

Solution (15 points)

a) The second derivative matrix is

$$A_1 = \begin{bmatrix} 3x^2 + 2y & 2x \\ 2x & 2 \end{bmatrix}$$

For this matrix to be positive definite, we need both top-left determinants to be positive. Thus $3x^2 + 2y > 0$ and $6x^2 + 4y - 4x^2 = 2(x^2 + 2y) > 0$. In this case the second condition implies the first, so we need $x^2 + 2y > 0$.

Answer to interesting question: we can rewrite $f_1 = \frac{1}{4}(x^2+2y)^2$. So, f_1 is always non-negative, and is 0 only along the curve $x^2 + 2y = 0$. So these are all "minimal points" in a sense. However, the function can't be strictly concave upwards around these points (because it looks flat along $x^2 + 2y = 0$), so it can't be positive definite here. This curve is also the locus where both partial derivatives vanish.

b) The second derivative matrix is

$$A_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$$

The top-left determinants are $6x$ and -1. So this matrix is never positive definite.

Answer to interesting question: Suppose both partial derivatives vanish. That is, we look at the points where $3x^2 + y - 1 = 0$ and $x = 0$. Combining these gives only the point $(0, 1)$. At this point the eigenvalues are 1 and -1, so that we get a saddle point.

Problem 7: Do problem 3 in section 8.1 (pg. 410).

Solution (15 points)

We are asked to find when the equation $A^TCAu = f$ is solvable, i.e. what is the column space of A^TCA? Here, the matrix from equation (9) is

$$A^TCA = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

Since the middle column is the sum of the outer two with the sign switched, we see that the column space has basis $[1, -1, 0]^T$ and $[0, -1, 1]^T$. Recall that the equations giving the column space are determined by the perpendicular vectors. In this case the perpendicular vectors is $[1, 1, 1]^T$, so that $C(A^TCA)$ is the set of all vectors whose components add to 0. Physically speaking, this means that we are looking at the case when the entire apparatus has no net force on it - if our equations are satisfied then the only forces are coming from the springs.

When the forces are $(-1, 0, 1)$, one solution is $u = (\frac{1}{c_2}, 0, \frac{-1}{c_3})$. That is, the middle mass is fixed, and the two end masses are vibrating opposite each other. The complete solution is a particular solution plus the nullspace. Since A^TCA is symmetric, the nullspace is perpendicular to the column space, i.e. the nullspace has basis $[1, 1, 1]$. So the complete solution is $u = (m + \frac{1}{c_2}, m, m + \frac{-1}{c_3})$. Physically speaking, this means that every solution is given by the particular one we found after shifting the whole apparatus upwards or downwards.

Problem 8: Do problem 11 in section 8.1 (pg. 411). This problem requires Matlab.

Solution (15 points) (Thanks to Peter Buchak.)

This problem asks for you to find numerical solutions of the differential equation

$$-\frac{d^2u}{dx^2} + 10\frac{du}{dx} = 1$$

with boundary conditions $u(0) = 0$ and $u(1) = 0$.

We're told $\Delta x = 1/8$, which means we're going to find the solution $u(x)$ at points spaced 1/8 apart, that is, $u(1/8)$, $u(2/8)$, ..., $u(7/8)$. These unknowns will be stored in a vector $u = (u(1/8), u(2/8), ...u(7/8))$. (We leave off the two end measurements to represent that the boundary conditions are 0.) Since this vector has 7 components, the matrices in this problem will be 7 by 7.

du/dx can be approximated by multiplying by either the forward difference matrix or the backward difference matrix; this problem asks you to try both. $-d^2u/dx^2$ can be approximated by multiplying by the second difference matrix. The Matlab code copied below shows one way to construct these matrices. (I called them `F`, `B`, and `S`.) You could also type them in by hand. When you run the code, it will display the matrices so you can see what they look like. They should resemble the matrix in equation (13) of section 8.1 of the textbook, for example.

The right hand side of the equation is just the function $f(x) = 1$, which at the 7 points is just a vector of 7 ones.

To solve the equation, combine the matrices for the second derivative and the first derivative to get a single matrix K that performs the operation $-d^2/dx^2 + 10d/dx$. Then solve the linear system $Ku = f$ with the Matlab command `u = K\f`. The Matlab code does this twice, with a K matrix that uses the forward difference (`K1`) and a K matrix that uses the backward difference (`K2`), to get two approximate solutions, `u1` and `u2`. Both of these are plotted, along with the exact solution. (For this simple equation, the formula for the exact solution could be obtained by the standard techniques of 18.03 for example.)

If you want, you can modify the `N=7` line of the code to solve for more than 7 unknowns (like 50 or 100 for example), to see how the approximate solutions get more accurate with more unknowns.

Brian's Remark: for the numbers to work out correctly, you need the second derivative matrix to have the same number everywhere along the diagonal. If you used $A'A$ for the 7 by 7 matrix A, then the bottom right hand will be incorrect. However, if you use the 9 by 9 matrix A, calculate $A'A$, and then cut off the first and last columns and rows, you do get the correct matrix. Sorry about the confusion - I've instructed the graders to give full credit for this.

```
function problem8 % problem set 8 section 8.1 problem 11

N=7; % 7 unknowns
dx=1/(N+1); % dx=1/8

% construct forward, backward, and second difference matrices
F=(diag(ones(1,N-1),1)-diag(ones(1,N)))/dx
B=(diag(ones(1,N))-diag(ones(1,N-1),-1))/dx
S=(2*diag(ones(1,N))-diag(ones(1,N-1),1)-diag(ones(1,N-1),-1))/(dx^2)
f=ones(N,1); % right hand side

% solve K*u=f using both forward and backward differences
K1=S+10*F
u1=K1\f;
K2=S+10*B
u2=K2\f;
```

```
% plot these solutions, along with exact solution for comparison
x=(1:N)/(N+1);
plot(x,u1,'+k'); % forward difference solution: +'s
hold on
plot(x,u2,'xk'); % backward difference solution: x's
x3=0:.01:1;
u3=1/(10*(exp(10)-1))*(1+(exp(10)-1)*x3-exp(10*x3));
plot(x3,u3,'k');
hold off
grid on
```

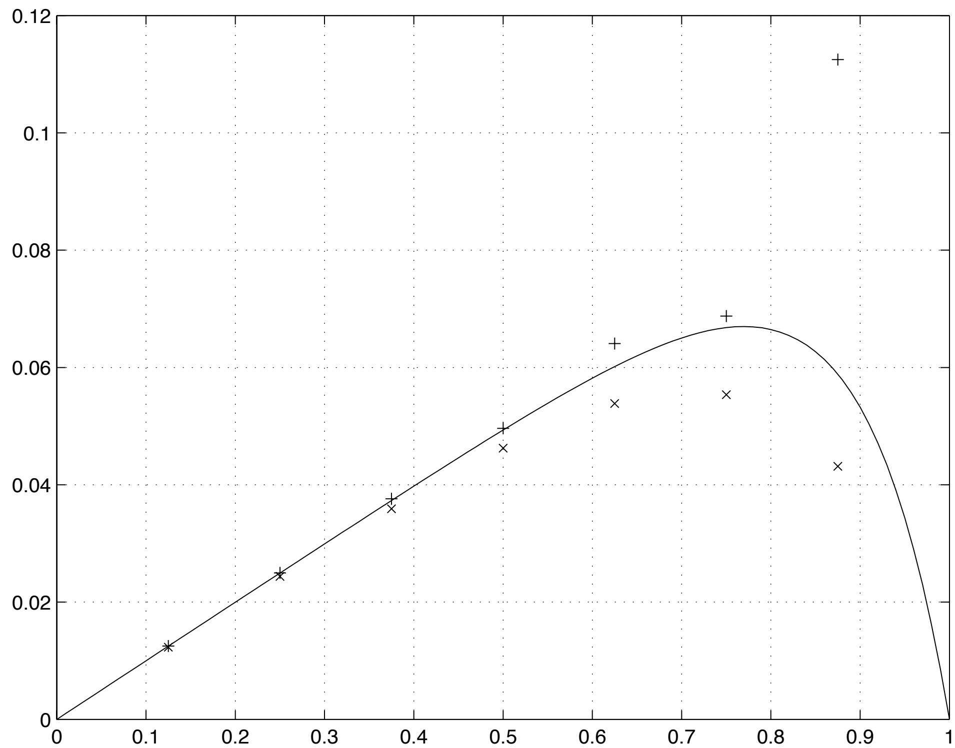

Figure 1: Approximate solutions using forward (+) and backward (x) differences, along with exact solution (solid line).