# 18.06 Problem Set 7

Due Wednesday, 16 April 2008 at 4 pm in 2-106.

**Problem 1:** Do problem 1 in section 6.3 (pg. 315) in the book.

Solution (10 points)

We solve a linear system of differential equations by taking

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

where $\lambda_1, \lambda_2$ are the eigenvalues, $x_1, x_2$ are the eigenvectors, and $c_1, c_2$ are constants that satisfy $c_1 x_1 + c_2 x_2 = u(0)$.

To write down the matrix exponential explicitly, we must find the eigenvalues and eigenvectors of $A$. Since this $A$ is diagonal, its eigenvalues are just the diagonal entries, i.e. $\lambda_1 = 4$ and $\lambda_2 = 1$. The eigenvectors are $x_1 = (1, 0)$ and $x_2 = (1, -1)$. Finally, if $u(0) = (5, -2)$ we must find how to write $u(0)$ as a linear combination of $x_1$ and $x_2$. We do this by solving the equation

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

We get $c_2 = 2$ and $c_1 = 3$. So, the final equation is

$$u(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

**Problem 2:** Do problem 3 in section 6.3 (pg. 315).

Solution (10 points)

To linearize this system, we identify $u$ with the vector $[y, y']^T$, so that we have two equations $dy/dt = y'$ and $dy'/dt = y'' = 4y + 5y'$. That is, we can "decouple" the differential equation by adding $y'$ as a new variable, to obtain the system

$$\begin{bmatrix} dy/dt \\ dy'/dt \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

We call the coefficient matrix $A$ as usual. The eigenvalues of $A$ satisfy the equation $\lambda^2 - 5\lambda - 4 = 0$, so the eigenvalues are $\lambda_1 = \frac{1}{2}(5 + \sqrt{41})$ and $\lambda_2 = \frac{1}{2}(5 - \sqrt{41})$.

Another way to find the eigenvalues is to substitute $y = e^{\lambda t}$ into the differential equation. We obtain

$$\lambda^2 e^{\lambda t} = 5\lambda e^{\lambda t} + 4e^{\lambda t}$$

Dividing by $e^{\lambda t}$, we find the same relationship $\lambda^2 - 5\lambda - 4 = 0$.

**Problem 3:** a) Do problem 17 in section 6.3 (pg. 317).

b) Do problem 24 in section 6.3 (pg. 318).

Solution (5+5 points)

a) The infinite series for $e^{Bt}$ is

$$e^{Bt} = I + tB + \frac{1}{2}t^2B^2 + \ldots$$

However, since $B^2 = 0$, all the terms of this sequence will be zero except for the first two. Thus

$$e^{Bt} = I + tB = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

The derivative is

$$d(e^{Bt})/dt = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Of course, this is the same thing as $Be^{Bt}$ (just multiply it out).

b) First, recall that $e^{A+B} = e^A e^B$ whenever $AB = BA$. The matrices $A$ and $-A$ always commute (both products are $-A^2$), so $e^{At}e^{-At} = e^0 = I$. Thus $e^{At}$ is always invertible. You could also check this by multiplying out the power series formally.

Second, we know that $e^{At}$ has diagonalization $Se^{\Lambda t}S^{-1}$. That is, the eigenvalues of $e^{At}$ are just $e^{\lambda t}$ for eigenvalues $\lambda$ of $A$. However, $e^{\lambda t}$ is never 0, so $e^{At}$ never has 0 for an eigenvalue, meaning that it is always invertible.

**Problem 4:** a) Do problem 4 in section 6.4 (pg. 327).

b) Do problem 10 in section 6.4 (pg. 327).

Solution (5+5 points)

a) We need to diagonalize $A$; since $A$ is symmetric, we know that we will be able to pick perpendicular eigenvectors. If we normalize these eigenvectors to length 1, the eigenvector matrix will be orthogonal. $A$ has eigenvalues given by the equation

$\lambda^2 - 5\lambda - 50 = 0$, so $A$ has eigenvalues $\lambda_1 = 10$ and $\lambda_2 = -5$. The corresponding eigenvectors of unit length are $x_1 = \frac{1}{\sqrt{5}}[1,2]^T$ and $x_2 = \frac{1}{\sqrt{5}}[-2,1]^T$. So

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

b) The flaw here is that $x^T x$ is not necessarily a real number (and neither is $x^T Ax$). We know that $x^H x$ is always real, since it is the length of $x$ squared. But in general $x^T x$ is not real - take for example the one-component vector $x = [1 + i]$.

**Problem 5:** Do problem 15 in section 6.4 (pg. 328).

Solution (10 points)

We diagonalize

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$$

The eigenvalues are given by the equation $\lambda^2 = 0$, so the only eigenvalue is $\lambda = 0$. The eigenvectors are then given by the nullspace of $A$. We find this using row reduction, just as for real matrices:

$$\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$$

The nullspace is one dimensional, meaning that every eigenvector is a multiple of $[1, -i]^T$.

**Problem 6:** Do problem 24 in section 6.4 (pg. 329).

Solution (10 points) (No justifications necessary.)

We start with $A$. It is definitely invertible. It is orthogonal since $P^T = P^{-1}$ (both are equal to $P$). It is not a projection matrix because $P^2 \neq P$. It is clearly a permutation matrix. It is diagonalizable because it is symmetric (so that we can find a basis of orthonormal eigenvectors). It is Markov because all entries are non-negative and the columns add to 1.

$A$ does not have an $LU$-decomposition, because we must do a row swap in reducing $A$. It does have a $QR$-decomposition because the columns are linearly independent. It is diagonalizable, so it has an $S\Lambda S^{-1}$ decomposition. Because it is also symmetric, the diagonalization actually gives a $Q\Lambda Q^T$ decomposition.

$B$ is not invertible (it has rank 1). It is not orthogonal as it has no inverse. It is a projection matrix, because $B^2 = B$ and $B^T = B$ - in fact it projects onto the vector $(1, 1, 1)$. It is not a permutation matrix. It is diagonalizable because it is symmetric. It is Markov.

$B$ does have an $LU$-decomposition, since we do not need a row swap. It doesn't have a $QR$-decomposition because the columns are dependent. It is diagonalizable and symmetric, so it has both a $S\Lambda S^{-1}$ and a $Q\Lambda Q^T$ factorization.

**Problem 7:** Do problems 3 and 4 from section 10.2 (pg. 492).

Solution (10 points)

We solve the equation $Az = 0$ by reducing:

$$\begin{aligned}\begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} &\rightsquigarrow \begin{bmatrix} i & 1 & i \\ 0 & 2i & i-1 \end{bmatrix} \\ &\rightsquigarrow \begin{bmatrix} 1 & -i & 1 \\ 0 & 1 & (i-1)/2i \end{bmatrix} \\ &\rightsquigarrow \begin{bmatrix} 1 & 0 & (i+1)/2 \\ 0 & 1 & (i+1)/2 \end{bmatrix}\end{aligned}$$

This matrix has one free column, so we get one special solution

$$(-(i+1)/2, -(i+1)/2, 1)$$

I'll rescale to use $z = (i+1, i+1, -2)$ instead, it doesn't make any difference. The matrix $A^H$ is

$$A^H = \overline{A}^T = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix}$$

Column 1 is $[-i, 1, -i]^T$. To calculate $C_1 \cdot z$, we need to take

$$C_1 \cdot z = C_1^H z = [i, 1, i] \begin{bmatrix} i+1 \\ i+1 \\ -2 \end{bmatrix} = 0$$

Similarly,

$$C_2 \cdot z = C_2^H z = [1, i, i] \begin{bmatrix} i+1 \\ i+1 \\ -2 \end{bmatrix} = 0$$

Of course these equations must be true; by taking the Hermitian of a column of $A^H$, we are just getting a row of $A$, and we know that any row of $A$ times a vector in the nullspace gives 0.

The matrix $A^T$ is

$$A^T = \begin{bmatrix} i & 1 \\ 1 & i \\ i & i \end{bmatrix}$$

These columns are not perpendicular to $z$, for example

$$C_1 \cdot z = C_1^H z = [-i, 1, -i] \begin{bmatrix} i+1 \\ i+1 \\ -2 \end{bmatrix} = 2 + 2i$$

Putting all this together, we see that the four fundamental spaces should be $C(A)$, $N(A)$, $C(A^H)$ and $N(A^H)$. They will satisfy the same orthogonal relationships as before: $N(A)$ and $C(A^H)$ are orthogonal complements, and $C(A)$ and $N(A^H)$ are orthogonal complements.

**Problem 8:** Do problem 15 from section 10.2 (pg. 493).

Solution (10 points)

Since $A$ is Hermitian, we expect it to have real eigenvalues, and a unitary eigenvector matrix $U$.

$A$ has eigenvalues given by the equation $\lambda^2 - \lambda - 2 = 0$, so we find eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The corresponding normalized eigenvectors are $x_1 = \frac{1}{\sqrt{6}}[1-i, 2]^T$ and $x_2 = \frac{1}{\sqrt{3}}[i-1, 1]^T$. (Another choice in the same direction is $\frac{1}{\sqrt{6}}[-2, 1+i]^T$, which is more symmetric-looking.) So we obtain

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$U = \frac{1}{\sqrt{6}} \begin{bmatrix} 1-i & -2 \\ 2 & 1+i \end{bmatrix}$$

Since the columns of $U$ are (complex) orthogonal unit vectors, $U$ is unitary.

**Problem 9:** Do problem 6 in section 10.3 (pg. 500).

Solution (10 points)

The Fourier matrix $F_4$ is

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

We can multiply this out to find

$$F_4^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

and

$$F_4^4 = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

**Problem 10:** Do problem 11 in section 10.3 (pg. 501).

Solution (10 points)

Multiplying the two given matrices, we find that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = i^2 = -1$, and $\lambda_1 = i^3 = -i$.